Notes on Relativistic Quantum Field Theory: A course given by Dr. Tobias Osborne

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June 24, 2021

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1 Lecture 1: Motivations for (Relativistic) Quantum Field Theory

We assume that there is a "Theory of Everything", called ToE herein, which is suspected to include a contained description of small and large scale forces in the universe (e.g., strong nuclear, weak nuclear, electromagnetic, and gravitational forces). Our classical observations fall into a low energy, large scale, decoherent approximation of this ToE. The purpose of this collection of notes is to study *(effective) relativistic quantum field theory* which is the low energy, large scale approximation of the ToE, and we are not sure how many "steps" there are in between the two, but the relativistic quantum field theory is the closest we currently come to a ToE. If we lower the energy and lengthen the scale of our study, we land in a nonrelativistic quantum field theory. A quantum field theory is then subject to decoherence, as we exist on the classical scale, and we begin the development of a relativistic quantum field theory with a classical field theory. In this sense, our purpose is to undo all of the approximations that nature forces us to take when we run experiments, take measurements, and devleop theoretical frameworks to describe observed phenomena.
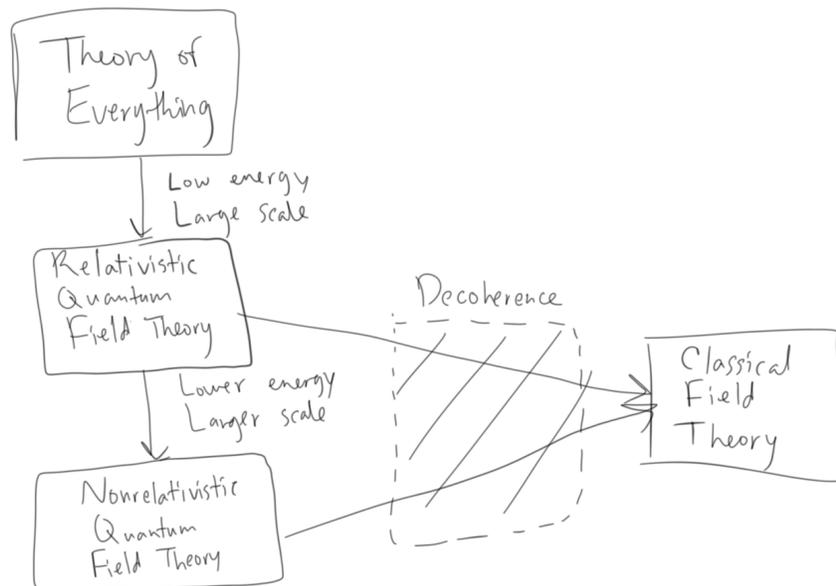

Figure 1: Schematic of the study of field theories in physics.

Mathematical Machinery of Relativistic QFT

For a quantum field theory to be relativistic it must be symmetric under the Poincaré group transformations.

Let the four-vector (x_0, x_1, x_2, x_3) be the spacetime coordinates in an inertial reference frame. Then in any other reference frame that an observer chooses, the following condition is satisfied.

(Note: We adopt the Einstein summation notation, and repeated indices are summed over, such that μ, ν, ρ, and σ below are summed from 0 to 3.)

$$\eta_{\mu\nu} dx'^{\mu} dx'^{\nu} = \eta_{\rho\sigma} dx^{\rho} dx^{\sigma} \tag{1}$$

Where we are using the Minkowski metric

$$\eta_{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{2}$$

Any transformation satisfying the first equation, the relationship between the coordinates of two reference frames, must be a *linear transformation* of the form, denoted by the pair (Λ, a),

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu} \tag{3}$$

Where $\Lambda^{\mu}{}_{\nu}$ is the $4x4$ Lorentz transformation matrix that represents rotations, and a^{μ} is a constant four-vector that represents spatial translations.

Also note that the Lorentz transformation must satisfy the following condition, shown in index and matrix notation.

$$\eta_{\mu\nu} \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} = \eta_{\rho\sigma} \tag{4}$$

$$\Lambda^T \eta \Lambda = \eta \tag{5}$$

These transformations (Λ, a) are the elements of the Poincaré group $\mathcal{P}_4$. Let's check the group conditions, using equation (**??**) for calculating the product of two Poincaré transformations,

$$
\begin{array}{llll}
product & (\Lambda, a) \circ (\bar{\Lambda}, \bar{a}) & = & (\bar{\Lambda}\Lambda, \bar{\Lambda}a + \bar{a}) \\
identity & (1, 0) \circ (\Lambda, a) & = & (\Lambda, a) \\
inverse & (\Lambda^{-1}, a^{-1}) \circ (\Lambda, a) & = & (1, 0) \\
associativity & \left((\Lambda, a) \circ (\bar{\Lambda}, \bar{a})\right) \circ (\bar{\bar{\Lambda}}, \bar{\bar{a}}) & = & (\Lambda, a) \circ \left((\bar{\Lambda}, \bar{a}) \circ (\bar{\bar{\Lambda}}, \bar{\bar{a}})\right)
\end{array}
\tag{6}
$$

In index notation, the total effect of the product of two Poincaré transfomations is written as

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu} \tag{7}$$

$$x''^{\mu} = \bar{\Lambda}^{\mu}{}_{\rho} x'^{\rho} + \bar{a}^{\mu} \tag{8}$$

Quantum mechanical symmetries are representated by *unitary* or *anti-unitary* linear operators as elements of a separable Hilbert space $\mathscr{H}$ such that for each Poincaré transformation $(\Lambda, a) \in \mathcal{P}_4$, there exists a unitary transformation

$$U(\Lambda, a) : \mathscr{H} \to \mathscr{H} \tag{9}$$

The identity unitary transformation, and the product of any two unitary transformations of elements of the Poincaré group are physically indistinguishable from each other up to a phase factor $e^{i\phi}$, where $\phi \in \mathbb{R}$

$$U(\bar{\Lambda}, \bar{a})U(\Lambda, a) = e^{i\phi((\Lambda, a),(\bar{\Lambda}, \bar{a}))}U(\bar{\Lambda}\Lambda, \bar{\Lambda}a + \bar{a}) \tag{10}$$

$$U(\mathbb{1}, 0) = e^{i\phi}\mathbb{1}. \tag{11}$$

The family of transformations $U(\Lambda, a)$ that satisfy these equations is called the *projective unitary representations of* $\mathcal{P}_4$, and is precisely what we need to establish a relativistic quantum field theory that includes translations, rotations, and Lorentz boosts as its transformations.

Consider the *time translation subgroup* of the Poincaré group

$$\{(\Lambda = \mathbb{1}, a = (t, 0, 0, 0)) : t \in \mathbb{R}\} \subset \mathcal{P}_4 \tag{12}$$

And let U be a unitary representation of $\mathcal{P}_4$. Then $V(t) = U(\mathbb{1}, (t, 0, 0, 0))$ is a one-parameter family of unitary transformations, which is a group homomorphism, such that $V(s)V(t) = V(s + t)$, and is called a *propagator*, and is a solution to the Schrödinger equation, assuming the Hamiltonian $\hat{H}$ is self-adjoint and is stable, such that there at most a finite number of, preferably zero, negative eigenvalues

$$\frac{dV(t)}{dt} = i\hat{H}V(t). \tag{13}$$

So, having a unitary representation is equivalent to solving the time-dependent Schrödinger equation, and we require that $U(\Lambda, a)$ is positive energy, such that the spectrum of eigenvalues of $\hat{H}$ is positive $spec(\hat{H}) \subset \mathbb{R}^+$, and stable. Otherwise, the system may be unstable and plunge to more and more negative eigenvalues.

All single-particle unitary representations of $\mathcal{P}_4$ have been classified by Wigner, and are labelled by their mass m (invariant under Poincaré and Lorentz transformations) and their helicity/spin s. The bad news is that the universe is comprised of many particles, creating tension between locality and interactions, rendering thorough study of single-particle representations pointless. Even for a single particle, as we increase locality, self-interaction increases, and new particles are created via tunnelling.

Why is Obtaining a Relativistic Quantum Field Theory Hard?

The reason that constructing a relativistic QFT is considered so difficult is because there are no nontrivial, only trivial, finite-dimensional unitary representations of the Poincaré group $\mathcal{P}_4$. The only nontrivial representations of $\mathcal{P}_4$ are infinite-dimensional, and are not easy to work with when constructing a unitary representation. Consider this in contrast to the, more easily constructed, special orthogonal group of three-dimensional rotations $SO(3)$ with unitary representations

$$U = e^{i(\sigma_x \hat{J}_x + \sigma_y \hat{J}_y + \sigma_z \hat{J}_z)}. \tag{14}$$

Where $\hat{J}_\alpha$ are the angular momentum operators. The reason $SO(3)$ is easier to work with and construct a unitary representation is because $SO(3)$ a *compact* group, while $\mathcal{P}_4$ is a *non-compact* group.

2 Lecture 2: Introduction to Classical Field Theory

To build a relativistic QFT, we start with an effective model from a *classical field theory*, and make an "educated guess" to quantize the classical field theory. The desired relativistic QFT has nothing to do *a priori* with the classical field theory. After quantization, the "educated guess", we take the limits of low energy, large scale, and decoherence, and check that we get back the classical field theory we started with, demonstrating whether the chosen quantization is correct or incorrect... enough, to some level of approximation.

Fields

A *field* is a quantity (e.g., density, spin, charge) tht is defined at every point on a manifold $\mathcal{M}$. Note that a rigorous definition of a field requires the introduction of vector bundles, of which we will not go so far.

We work on the Minkowski spacetime manifold $\mathcal{M} = \mathcal{M}_{1,3} = \mathbb{R}^1 \times \mathbb{R}^3$, with the field often taken to be two-times differentiable such that $\phi \in C^2(\mathcal{M}, \rho)$ and defined as a function from the manifold to some target space ρ

$$\phi : \mathcal{M} \to \rho \tag{15}$$

Some target spaces ρ, their associated field type, and example applications and models include

- $\rho = \mathbb{R}$; **scalar field**; charge density, magnetization density, Higgs boson

- $\rho = \mathbb{R}^n$; **vector field**; electromagnetic field (actually a gauge field), pions
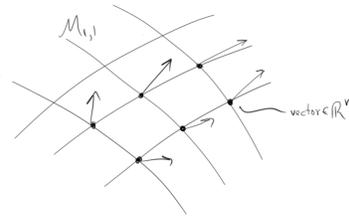
Figure 2: Sketch of the vector field $\rho = \mathbb{R}^n$ over the Minkowski space $\mathcal{M}_{1,1}$.

- $\rho = \mathcal{S}^2$; **vector field on the surface of a sphere**; σ-model, quantum magnets

- $\rho = \mathcal{S}^1 \times \mathcal{S}^1$; **vector field on a torus**; Chern-Simons theory, Lie groups

Note that when our target space is the N-dimensional real vector field $\mathbb{R}^N$, the vector field is described as a list N scalar fields $\{\phi_a(x)\}_{a=1}^N$, where x is the coordinate four-vector.

Dynamics of Classical Fields

We restrict this to discussion to classical dynamics generated by Lagrangians, obtained via the variational principle applied to th action functional, where the system of scalar fields $\{\phi_a(x)\}_{a=1}^N$, and a labels the particle type (e.g., charge). The action functional $\mathcal{S}$ contains a function of the Langrangian density $\mathscr{L} = \mathscr{L}(\phi_a, \partial_\mu \phi_a)$. The Lagrangian density is actually a function of higher order derivatives of the fields ϕ_a, but we make the assumption and approximation of first order derivatives, based on observation

$$\mathcal{S}(\Omega) = \int_\Omega d^4x \; \mathscr{L}(\phi_a, \partial_\mu \phi_a). \tag{16}$$

Where $d^4x = dx_0 dx_1 dx_2 dx_3$ and $\Omega \subset \mathcal{M}_{1,3}$, as a measurable set, is a region in $(3+1)$-dimensional spacetime. Typically, we consider the spacetime region as the entire Minkowski space $\Omega = \mathcal{M}_{1,3}$.

To extract the equations of motion, we suppose that the action $\mathcal{S}$ is stationary under infinitesimal variations of the component scalar fields $\phi_a(x) \to \phi_a(x) + \delta\phi_a(x)$, which vanish on the spacetime region boundary, such that $\delta\phi_a(x) = 0$ on $\partial\Omega$.

Varying the action functional, we obtain N *Euler-Lagrange equations of motion*

$$\delta\mathcal{S}(\Omega) = \int_\Omega d^4x \; \left(\frac{\partial \mathscr{L}}{\partial \phi_a} \delta\phi_a + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_a)} \delta(\partial_\mu \phi_a) \right) \tag{17}$$

$$0 = \int_\Omega d^4x \; \left(\frac{\partial \mathscr{L}}{\partial \phi_a} \delta\phi_a - \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_a)} \right) \right) \delta\phi_a + \int_\Omega d^4x \; \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_a)} \delta\phi_a \right) \tag{18}$$

$$0 = \int_\Omega d^4x \; \left(\frac{\partial \mathscr{L}}{\partial \phi_a} \delta\phi_a - \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_a)} \right) \right) \delta\phi_a + \partial_\mu \mathcal{M}^\mu. \tag{19}$$

The last term is a surface term which vanishes on the boundary, since we first demanded that $\delta\phi_a = 0$ vanishes on the boundary, such that $\partial_\mu \mathcal{M}^\mu = 0$ on $\partial\Omega$.

Since $\mathcal{S}(\Omega)$ is stationary with respect to all variations of the fields $\delta\phi_a$ and admissable spacetime regions Ω, the integrand of the remaining term must also vanish $\forall\, a = 1, 2, \ldots, N$, and we obtain the N Euler-Lagrange equations of motion

$$-\frac{\partial \mathscr{L}}{\partial \phi_a} + \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_a)} \right) = 0. \tag{20}$$

So, with equations of motion in hand, gotten by whatever means, they can be encoded in the Langrangian density, and then recovered via the Euler-Lagrange equations of motion. This allows us to discover equations of motion with certain properties, such as being symmetric under the Poincaré transformations, by designing Lagrangian densities, which are scalars under these symmetry transformations, apply Euler-Lagrange "recipe", and get the equations of motion,

which are guaranteed to be symmetric under the chosen transformations. This benefit of the action principle makes it easy to design equations of motion with certain symmetries.

Example: Klein-Gordon field

Consider the Langrangian density

$$\mathscr{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2 \tag{21}$$

$$= \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 \tag{22}$$

Where $\dot{\phi} = \partial_0\phi$ is the time derivative of the field, and $(\nabla\phi)_j = \partial_j\phi$ for $j = 1, 2, 3$ are the x, y, z spatial components of the field derivatives.

Apply the Euler-Lagrange equations of motion to obtain the dynamics of the Klein-Gordon field

$$-\frac{\partial\mathscr{L}}{\partial\phi_a} + \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_a)}\right) = 0 \tag{23}$$

$$\frac{\partial}{\partial x^\mu}(\partial^\mu\phi) - (-m^2\phi) = 0 \tag{24}$$

$$\partial_\mu\partial^\mu\phi + m^2\phi = 0 \tag{25}$$

$$\Box\phi + m^2\phi = 0 \tag{26}$$

Hamiltonian formalism

A better way to guess/build a quantum theory, with the correct calssical limit determined by the Langrangian density, is the *Hamiltonian formalism*, where we calculate conjugate variables and impose canonical (algebraic) commutation relations.

Suppose that $\phi_a(x)$ is a component field's *canonical position*. Then define the *conjugate momentum density* for each field to be

$$\pi_a(x) = \frac{\partial\mathscr{L}}{\partial\dot{\phi}_a}. \tag{27}$$

In order to computationally study a field, we discretize the space into a regular lattice with spacing ϵ between each lattice site, since representing a continuous object on a computer would require an infinite amount of data. Discretization yields generalized coordinates, defined by the field itself, of the form $q_j^a(t) = \phi_a(t, x_j)$, $j \in \mathbb{Z}$. For example, consider the $1 + 1$-dimensional Minkowski space $\mathcal{M}_{1,1}$

Figure 3: Schematic of field discretization.

To discretize the action functional and apply the variational principle, replace the partial derivatives $\partial_\mu \phi_a(t, x)$, since it is a continuous operation, by applying Taylor's theorem to approximate the spatial component of the field as a finite difference, where ϵ should be made as small as possible such that the error approximation is minimized

$$\partial_x \phi_a(t, x_j) \cong \frac{\phi_a(t, x_j + \epsilon) - \phi_a(t, x_j)}{\epsilon} = \frac{q^a_{j+1} - q^a_j}{\epsilon}. \tag{28}$$

Leave the temporal component to be continuous

$$\partial_t \phi_a(t, x_j) \cong \dot{q}^a_j. \tag{29}$$

To obtain the Lagrangian, integrate the density over space, such that only the time dependence is left

$$\int d^3 x \, \mathscr{L}(\phi_a, \partial_\mu \phi_a) = L(q^a_j(t), \dot{q}^a_j(t)) = L(t). \tag{30}$$

The discrete approximation of the Lagrangian is written as

$$L(t) \cong \sum_j \delta x_j \mathscr{L}(\phi_a(t, x_j), \partial_\mu \phi_a(t, x_j)). \tag{31}$$

And the discrete conjugate momenta

$$p^a_j(t) = \frac{\partial L}{\partial \dot{q}^a_j} = \sum_j \delta x_j \frac{\partial \mathscr{L}}{\partial \dot{q}^a_j} = \sum_j \delta x_j \pi^a(t, x_j). \tag{32}$$

Simple Example

As a simple example, consider the Lagrangian density

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \phi \partial^\mu \phi) \tag{33}$$

$$= \frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\partial_x \phi)^2. \tag{34}$$

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(RECHECK) Integrate over space d^3x to obtain the Lagrangian, and discretize using the rules defined above

$$L(t) = \frac{1}{2} \sum_{j=-\infty}^{\infty} \left(\epsilon \left(\frac{dq_j}{dt} \right)^2 - \frac{(q_{j+1} - q_j)}{\epsilon} \right). \tag{35}$$

Where $\delta x_j = \epsilon$ and $p_j(t) = \epsilon \frac{dq_j}{dt} = \pi(t, x_j)\delta x_j$.
As $\epsilon \to 0$, $q_j(t) \to \phi(t, x)$ and $p_j(t) \to \dot{\phi}(t, x)$

Hamiltonian Density

For the discrete approximation, the Hamiltonian H, obtained by integrating the Hamiltonian density $\mathscr{H}$ of space d^3x, is written as

$$H = \sum_j p_j^a \dot{q}_j^a - L = \sum_j \delta x_j (\pi_a(t, x_j)\dot{\phi}_a(t, x_j) - \mathcal{L}_j) \tag{36}$$

In the limit as $\epsilon \to 0$, the Hamiltonian density is

$$\mathscr{H}(t, x) = \pi_a(t, x)\dot{\phi}_a(t, x) - \mathscr{L}(\phi_a(t, x), \partial_\mu \phi_a(t, x)) \tag{37}$$

The Hamiltonian density for the Klein-Gordon field is, dropping the field index and spacetime dependencies from the expression,

$$\mathscr{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \tag{38}$$

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3   Lecture 3: Symmetries in Classical Field Theory

Suppose that $\mathscr{L}(\phi_a, \partial_\mu \phi_a)$ is the Lagrangian density for some set of fields $\{\phi_a(x)\}_{a=1}^N$. Recall that the Lagrangian density is a compact, encrypted of writing the equations of motion of a system, and the Euler-Lagrange equations and the principle of least action are used to unpack/decrypt the equations of motion.

Now consider an infinitesimal continuous transformation of the fields

$$\phi_a'(x) = \phi_a(x) + X_a(\phi_a) \tag{39}$$

This produces an *infinitesimal symmetry* in the equations of motion when they are left invariant under the principle of least action

$$\mathscr{L} \to \mathscr{L}(\phi_a', \partial_\mu \phi_a') = \mathscr{L}(\phi_a, \partial_\mu \phi_a) + \partial_\mu F^\mu \tag{40}$$

$$\&\ \ \mathcal{S}[\phi_a] = \int d^4 x \mathscr{L} = \int d^4 x (\mathscr{L} + \partial_\mu F^\mu) \tag{41}$$

Noether's Theorem

Every continuous symmetry of a Lagrangian implies the existence of a conserved current $j^\mu(x)$.

Proof: Let $X_a[\phi_a] = \delta\phi_a$ be an arbitrary, infinitesimal change in each field, such that the infinitesimal change in the Lagrangian density is

$$\delta\mathscr{L}(\phi_a, \partial_\mu \phi_a) = \mathscr{L}(\phi_a + \delta\phi_a, \partial_\mu(\phi_a + \delta\phi_a)) - \mathscr{L}(\phi_a, \delta_\mu \phi_a) \tag{42}$$

$$\cong \frac{\partial \mathscr{L}}{\partial \phi_a} \delta\phi_a + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_a)} \delta(\partial_\mu \phi_a) \tag{43}$$

$$= \left(\frac{\partial \mathscr{L}}{\partial \phi_a} - \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_a)} \right) \right) \delta\phi_a + \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_a)} \delta\phi_a \right) \tag{44}$$

$$\delta\mathscr{L}(\phi_a, \partial_\mu \phi_a) = \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_a)} \delta\phi_a \right) \tag{45}$$

Where in line (14), we have kept only first order terms $\mathcal{O}(\delta\phi_a)$, and have used the Taylor expansion and added zero to get to line (15). To obtain line (16), note that the first term in line (15) is equal to zero, since $\phi_a(x)$ obey the Euler-Lagrange equations.

As defined above, for an infinitesimal transformation, call $\delta\phi_a = X_a[\phi_a]$ and for an infinitesimal symmetry, call $\delta\mathscr{L} = \partial_\mu F^\mu$.

$$\partial_\mu F^\mu = \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_a)} X_a[\phi_a] \right) \tag{46}$$

$$0 = \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_a)} X_a[\phi_a] - F^\mu \right) \tag{47}$$

Call the conserved quantity the conserved current

$$j^\mu(x) = \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_a)} X_a[\phi_a] - F^\mu \right). \tag{48}$$

Now, a conserved current implies the existence of a conserved charge. For any measurable region in our Minkowski space $V \subset \mathcal{M}$, define the integral of the time-like component of the current as

$$Q_V = \int_V d^3x \ j^0(x) \tag{49}$$

Take $V = \mathbb{R}^3$, and assume that the current vanishes at infinity, such that $j \to 0$ on the boundary ∂V, and take the time derivative

$$\frac{dQ_{\mathbb{R}^3}}{dt} = \int_{\mathbb{R}^3} d^3x \ \partial_0 j^0(x) \tag{50}$$

$$= - \int_{\mathbb{R}^3} d^3x \ \partial_k j^k \tag{51}$$

$$= - \int_{\partial \mathbb{R}^3} j ds \tag{52}$$

$$\frac{dQ_{\mathbb{R}^3}}{dt} = 0 \tag{53}$$

Example

Consider an active transformation of spacetime coordinates $x^\mu \to x^\mu - \epsilon^\mu$.
Then each field transforms as

$$\phi'_a(x^\mu) = \phi_a(x^\mu + \epsilon^\mu) \tag{54}$$

$$= \phi_a(x^\mu) + \epsilon^\nu \partial_\nu \phi_a(x^\mu) \tag{55}$$

And the Lagrangian density transforms as, yielding $4 \times 4 = 16$ equations from summing over μ and ν

$$\mathscr{L}(x'^\mu) = \mathscr{L}(x^\mu + \epsilon^\mu) \tag{56}$$

$$= \mathscr{L}(x^\mu) + \epsilon^\nu \partial_\nu \mathscr{L}(x^\mu) \tag{57}$$

Where the infinitesimal field transformation is $X_a[\phi_a] = \epsilon^\nu \partial_\nu \phi_a(x^\mu)$, and the infinitesimal symmetry of the Lagrangian density is $\partial_\mu F^\mu = \epsilon^\mu \partial_\mu \mathscr{L}(x^\mu)$

Consider the infinitesimal element ϵ with basis vector entries $[\hat{\nu}]^\mu{}_\nu = \delta^\mu{}_\nu$

$$\epsilon^\mu = \epsilon\hat{\nu}^\mu = \epsilon\{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\}. \tag{58}$$

Apply Noether's theorem to each of the 4 symmetry terms $\epsilon\hat{\nu}^\mu$, yielding 16 total terms that we assign as the elements $T^\mu{}_\nu$ of the *energy-momentum* or *stress-energy tensor*

$$T^\mu{}_\nu = j^\mu{}_\nu = \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_a)}\partial_\nu\phi_a - \delta^\mu{}_\nu\mathscr{L}. \tag{59}$$

Note that each of the νth columns of the energy-momentum tensor correspond to one of the four conserved currents and translation in each of the νth directions

$$\partial_\mu T^\mu{}_\nu = 0, \quad \forall \ \nu. \tag{60}$$

In a *closed* system, the corresponding conserved charges, from the columns (conserved currents) of the energy-momentum tensor, are the total energy and the momentum in each of the three spatial directions

$$E = \int d^3x \ T^{00} \tag{61}$$

$$p^j = \int d^3x \ T^{0j}. \tag{62}$$

Example of the Example: Klein-Gordon Field

Consider the Klein-Gordon Lagrangian density $\mathscr{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$. The energy-momentum tensor has elements of the form

$$T^\mu{}_\nu = \partial^\mu\phi\partial_\nu\phi - \delta^\mu{}_\nu\mathscr{L} \tag{63}$$

$$T^{\mu\nu} = \eta^{\nu\nu'}T^\mu{}_{\nu'} = \partial^\mu\phi\partial^\nu\phi - \eta^{\mu\nu}\mathscr{L}. \tag{64}$$

The corresponding conserved charges are

$$E = \int d^3x \ \mathscr{H}(x) = \int d^3x \ (\partial^0\phi\partial^0\phi - \frac{1}{2}\partial^0\phi\partial^0\phi + \frac{1}{2}m^2\phi^2) \tag{65}$$

$$p^j = \int d^3x \ \dot{\phi}\partial^j\phi. \tag{66}$$

How To Apply Noether's Theorem

1. Identify the continuous symmetry.

2. Calculate the change in the Lagrangian density.

3. Calculate the change in each of the fields.

4. Work out the conserved currents and charges.

Infinitesimal Lorentz Transformations

Consider the transformation $x^\mu \to \Lambda^\mu{}_\nu x^\nu$, where $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$, and $\omega^\mu{}_\nu$ is infinitesimal. Next, recall the following property of the group of Lorentz transformations, restricting the possible values for ω

$$\eta = \Lambda^T \eta \Lambda \tag{67}$$

$$\eta^{\mu\nu} = (\delta^\mu{}_\sigma + \omega^\mu{}_\sigma)(\delta^\nu{}_\tau + \omega^\nu{}_\tau)\eta^{\sigma\tau} \tag{68}$$

$$0 =_{\mathcal{O}(\omega)} \omega^{\mu\nu} + \omega^{\nu\mu} \tag{69}$$

This is a linear equation in ω, since we have kept only up to first order terms in ω, and tells us that ω is an *antisymmetric*, infinitesimal generator of Lorentz transformations with six independent variables which define six continuous symmetries and six conserved currents and charges.

$$\omega = \begin{pmatrix} 0 & -\alpha & -\beta & -\gamma \\ \alpha & 0 & -\delta & -\epsilon \\ \beta & \delta & 0 & -\kappa \\ \gamma & \epsilon & \kappa & 0 \end{pmatrix} \tag{70}$$

The action of this infinitesimal Lorentz transformation on the fields is

$$\phi_a(x) \to \phi'_a(x) = \phi_a(\Lambda^{-1}x) \tag{71}$$

$$= \phi_a((\delta - \omega)x) \tag{72}$$

$$= \phi_a(x^\mu - \omega^\mu{}_\nu x^\nu) \tag{73}$$

$$=_{\mathcal{O}(\omega)} \phi_a(x) - \omega^\mu{}_\nu x^\nu \partial_\mu \phi_a(x) \tag{74}$$

Showing that the symmetry is defined by the infinitesimals

$$\delta\phi_a = -\omega^\mu{}_\nu x^\nu \partial_\mu \phi_a \tag{75}$$

$$\& \quad \delta\mathscr{L} = -\omega^\mu{}_\nu x^\nu \partial_\mu \mathscr{L} = -\partial_\mu(\omega^\mu{}_\nu x^\nu \mathscr{L}) \tag{76}$$

$$\& \quad j^\mu{}_\omega = \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_a)} \omega^\rho{}_\nu x^\nu \partial_\rho \phi_a + \omega^\mu{}_\nu x^\nu \mathscr{L}. \tag{77}$$

(CHECK how j to $\mathcal{J}$) Applying Noether's theorem tells us that the six independent conserved currents $\partial_\mu(\mathcal{J}^\mu)^{\rho\sigma} = 0$, and conserved charges, are of the form

$$(\mathcal{J}^\mu)^{\rho\sigma} = x^\rho T^{\mu\sigma} - x^\sigma T^{\mu\rho} \tag{78}$$

$$Q^{jk} = \int d^3x \; (x^j T^{0k} - x^k T^{0j}) \tag{79}$$

$$Q^{0j} = \int d^3x \; (x^0 T^{0j} - x^j T^{00}) \tag{80}$$

Call Q^{jk} the generators of rotations, and Q^{0j} the generators of boosts of the Lorentz transformations.

Generators

Let f and g be maps from phase space to the real numbers

$$f, g : \ \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}. \tag{81}$$

Define the Poisson bracket with the pairs of canonical coordinates (q_j, p_j)

$$\{f, g\} = \sum_{j=1}^{N} \left(\frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_j} \right) \tag{82}$$

$$\{f, H\} = \frac{df}{dt} \tag{83}$$

The field theory version of the Posson bracket is defined with the canonical coordinate pairs $(\phi(x), \pi(x))$

$$\{F, G\} = \int d^3x \ \left(\frac{\delta F}{\delta \phi(x)} \frac{\delta G}{\delta \pi(x)} - \frac{\delta F}{\delta \pi(x)} \frac{\delta G}{\delta \phi(x)} \right) \tag{84}$$

$$\{f, Q^{\rho\sigma}\} = \frac{\partial f}{\partial s^{\rho\sigma}} \tag{85}$$

Where the Poisson bracket of f and the conserved charges Q generate the corresponding symmetry transformations. Conserved cahrges also obey the Lie algebra obeyed by the Poincaré group.

Standard Dogma of Quantization

Basically, put hats on things

- Function f on phase space $\to$ linear operator $\hat{f}$ (observable) on Hilbert space

- Poisson bracket $\{f, g\} = h \to$ commutator $[\hat{f}, \hat{g}] = i\hat{h}$

- Conserved charge $Q^{\rho\sigma} \to$ conserved charge operator $\hat{Q}^{\rho\sigma}$

Where the conserved charge operators generate the Lorentz transformations on Hilbert space

$$\frac{d\hat{U}}{ds} = i[\hat{U}, \hat{Q}^{\rho\sigma}]\omega_{\rho\sigma} \tag{86}$$

4 Lecture 4: Field Quantization

More than one quantum field theory can have the same classical field theory as an effective model, making field quantization not a well-posed problem. Developing a quantum field theory is therefore built on educated guesses.

Canonical quantization of particles

The standard approach of *canonical quantization* is to begin with a classical theory and suppose n classical degrees of freedom, which are used to measure the canonical coordinate pairs, *position* q_j and *momentum* p_j, for each degree of freedom, such that the Poisson bracket is defined by $\{q_j, p_k\} = \delta_{jk}$. The total energy of the of the system is measured by the classical *Hamiltonian*, defined by

$$H = \sum_{j=1}^{n} \frac{p_j^2}{2m} + \frac{m}{2} \sum_{j,k=1}^{n} q_j [\mathbf{Q}]_{jk} q_k \tag{87}$$

Where $\mathbf{Q}$ is an $n \times n$ symmetric, positive matrix.

For example, consider the quantum harmonic oscillator, and take the naive approach by basically putting hats on everything. This ends up working for field quantization, and yields a unitary representation of the Poincaré group.

- **Canonical coordinates**: (q_j, p_j)
  $\rightarrow$ **Canonical coordinate operators** : $(\hat{q}_j, \hat{p}_j)$

- **Poisson bracket**: $\{q_j, p_k\} = \delta_{jk}$
  $\rightarrow$ **Commutator**: $[\hat{q}_j, \hat{p}_k] = i\delta_{jk}$

- **Hamiltonian**: H, as defined above
  $\rightarrow$ **Hamiltonian operator** : $\hat{H} = \sum_{j=1}^{n} \frac{\hat{p}_j^2}{2m} + \frac{m}{2} \sum_{j,k=1}^{n} \hat{q}_j [\mathbf{Q}]_{jk} \hat{q}_k$

To diagonalize the Hamiltonian operator, first note that since $\mathbf{Q}$ is a symmetric, positive $n \times n$ matrix, there exists an orthogonal matrix $\mathbf{O}$ (s.t., $\mathbf{O}^T \mathbf{O} = \mathbf{I}$), such that $\mathbf{O}\mathbf{Q}\mathbf{O}^T = \mathbf{D}$, where $\mathbf{D}$ is a diagonal matrix, where we call the diagonal elements $\{\omega_i^2\}_{i=1}^n$

Now, transform the canonical coordinates, using the orthogonal matrix, such

that the correct commutation relation is still obeyed

$$\hat{q}_j = \sum_{k=1}^{n} [\mathbf{O}]_{jk} \hat{q}'_k \tag{88}$$

$$\hat{p}_j = \sum_{k=1}^{n} [\mathbf{O}]_{jk} \hat{p}'_k \tag{89}$$

$$i\delta_{jk} = [\hat{q}'_j, \hat{p}'_k] \tag{90}$$

Using the facts that $\mathbf{O}^T = \mathbf{O}^{-1}$ and $\mathbf{O}\mathbf{Q}\mathbf{O}^T = \mathbf{D}$, the Hamiltonian becomes diagonalized

$$\hat{H} = \sum_{j=1}^{n} \frac{\hat{p'_j}^2}{2m} + \frac{m}{2} \sum_{j,k,l,m=1}^{n} \hat{q}'_l [\mathbf{O}^T]_{jl} [\mathbf{Q}]_{jk} [\mathbf{O}^T]_{km} \hat{q}'_m \tag{91}$$

$$\hat{H} = \sum_{j=1}^{n} \frac{\hat{p'_j}^2}{2m} + \frac{1}{2} \sum_{k=1}^{n} \omega_k^2 \hat{q'_k}^2 \tag{92}$$

$$\hat{H} = \frac{1}{2} \sum_{k=1}^{n} \omega_k (\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2}) \tag{93}$$

Where the annihilation and creation ladder operators that diagonalize the quantum harmonic oscillator Hamiltonian are defined as

$$\hat{a}_k = \sqrt{\frac{m\omega_k}{2}} (\hat{q}'_k + \frac{i}{m\omega_k} \hat{p}'_k) \tag{94}$$

$$\hat{a}_k^\dagger = \sqrt{\frac{m\omega_k}{2}} (\hat{q}'_k - \frac{i}{m\omega_k} \hat{p}'_k) \tag{95}$$

Canonical Quantization of Fields

To quantize the Klein-Gordon field, we follow the same seemingly naive approach of putting hats on everything. In this example fo quantizing a field, the continuous variable x is used, in contrast to the discrete labels j in the previous example of the quantum harmonic oscillator

- **Canonical coordinates**: $(\phi(x), \pi(x))$
  $\rightarrow$ **Canonical coordinate operators** : $(\hat{\phi}(x), \hat{\pi}(x))$

- **Poisson bracket**: $\{\phi(x), \pi(y)\} = \delta^{(3)}(x - y)$
  $\rightarrow$ **Commutator**: $[\hat{\phi}(x), \hat{\pi}(y)] = i\delta^{(3)}(x - y)$

    Note that this is the *equal time Poisson bracket*, such that $(x - y)$ is the spatial three-vector.

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Also note that this commutator is strange, as it is comprised of "two self-adjoint operators and something's that not even a function"

- **Hamiltonian**: $H_{KG} = \frac{1}{2} \int d^3x \ \left(\pi^2(x) + (\nabla \phi(x))^2 + \frac{1}{2} m^2 \phi^2(x) \right)$
  $\rightarrow$ **Hamiltonian operator** $: \hat{H}_{KG} = \frac{1}{2} \int d^3x \ \left(\hat{\pi}^2(x) + (\nabla \hat{\phi}(x))^2 + \frac{1}{2} m^2 \hat{\phi}^2(x) \right)$

Essentially, replace discrete sums with continuous integrals, by switching to a continuous label $j \rightarrow x$ and a continuous dynamical variable $q_j \rightarrow q_x = \phi(x)$. Solving the quantum Hamiltonian, by analogy of the canonical quantization of particles, should be as simple as creating the analog of the $\mathbf{Q}$ matrix and its diagonalization.

Replacing sums by integrals allows the full diagonalization of $\mathbf{Q}$, and, therefore, the full diagonalization of the Hamiltonian $\hat{H} = \hat{H}_{KG}$, but this does not yet yield a unitary representation of the Poincaré group or a valid relativistic quantum field theory. Diagonalizing the Hamiltonian only quantizes a one-parameter subgroup of the Poincaré group. The conserved currents, charges, and operators obeying the correct Lie algebra are still needed for a relativistic quantum field theory.

Diagonalization of the quantum field theory

The diagonalization of a field theory begins with emergence of the *Fourier transform*. Replace sums with integrals, and, since the matrix elements are described by two numbers, let's define a continuous function in two variables $K(x, y)$

$$\hat{q}_j = \sum_k [\mathbf{O}]_{jk} \hat{q}'_k \quad \rightarrow \quad \hat{\phi}(x) = \int d^3y \ K(x, y) \hat{\phi}(y) \tag{96}$$

Where $K(x, y)$ is the *kernel* of the Fourier transform.

Using the Fourier transform is motivated by certain features of symmetric matrices. Consider the *circulant* matrix, a type of Toeplitz matrix where each successive column is a cyclic permutation of the previous column, initialized by the first column vector, and has the form as an $n \times n$ matrix

$$\begin{bmatrix} c_0 & c_{n-1} & \dots & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & & c_2 \\ \vdots & c_1 & c_0 & \ddots & \vdots \\ c_{n-2} & & \ddots & \ddots & c_{n-1} \\ c_{n-1} & c_{n-2} & \dots & c_1 & c_0 \end{bmatrix}. \tag{97}$$

These matrices are diagonalized via the discrete Fourier transform, which is an $n \times n$ unitary matrix, though not orthogonal and may have complex entries

$$\mathbf{U} = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & \cdots & \\ 1 & \mu & \mu^2 & \cdots & \\ 1 & \mu^2 & \mu^4 & & \vdots \\ \vdots & \vdots & & \mu^{jk} & \\ & & \cdots & & \mu^{nn} \end{bmatrix} \tag{98}$$

Where $\mu = e^{\frac{2\pi i}{n}}$ is the nth roots of unity. The elements of the discrete Fourier transform, therefore, have the form $\frac{1}{\sqrt{n}} e^{\frac{2\pi i jk}{n}}$. Compare this to the continuous Fourier transform kernel function $K(x,y) = \frac{1}{2\pi} e^{ixy}$.

For transformations between position and momentum space, make the guess that the Fourier transform that will diagonalize our quantized Klein-Gordon Hamiltonian has the form

$$\hat{\phi}(x) = \int \frac{d^3p}{(2\pi)^3} e^{ip \cdot x} \hat{\phi}_p(p). \tag{99}$$

Where $\hat{\phi}_p(p)$ is the momentum space wavefunction, and is not Hermitian, such that $\hat{\phi}_p(p)^\dagger = \hat{\phi}_p(-p)$. To check if the guess is correct, apply $\hat{H}_{KG}$ to the transform defined above, and observe whether it is diagonalized or not.

As in the discrete case of diagonalization, we construct ladder operators

$$\hat{\phi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(\hat{a}_p e^{ip \cdot x} + \hat{a}_p^\dagger e^{-ip \cdot x} \right) \tag{100}$$

$$\hat{\pi}(x) = -i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{\omega_p}{2}} \left(\hat{a}_p e^{ip \cdot x} - \hat{a}_p^\dagger e^{-ip \cdot x} \right) \tag{101}$$

$$\omega_p = \sqrt{|p|^2 + m^2} \tag{102}$$

Check the commutation relation

$$[\hat{\phi}(x), \hat{\pi}(x')] = \frac{-i}{2} \int \frac{d^3p \, d^3p'}{(2\pi)^6} \sqrt{\frac{\omega_{p'}}{\omega_p}} \left([\hat{a}_{-p}^\dagger, \hat{a}_{p'}] - [\hat{a}_p, \hat{a}_{-p'}^\dagger] \right) e^{i(p \cdot x + p' \cdot y)} \tag{103}$$

$$\left([\hat{a}_p, \hat{a}_{p'}^\dagger] = (2\pi)^3 \delta^{(3)}(p - p') \cdot \mathbb{I} \right) \tag{104}$$

$$= i\delta^{(3)}(x - y) \tag{105}$$

Making this substitution, the quantum Klein-Gordon Hamiltonian is diagonalized

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$$\hat{H}_{KG} = \int d^3x \int \frac{d^3p d^3p'}{(2\pi)^6} e^{i(p+p')\cdot x} \left(\frac{-1}{4} \sqrt{\omega_p \omega_{p'}} (\hat{a}_p - \hat{a}^\dagger_{-p})(\hat{a}_{p'} - \hat{a}^\dagger_{-p'}) \right) \quad (106)$$

$$+ \frac{-pp' + m^2}{4\sqrt{\omega_p \omega_{p'}}} (\hat{a}_p + \hat{a}^\dagger_{-p})(\hat{a}_{p'} + \hat{a}^\dagger_{-p'})) \quad (107)$$

$$= \int \frac{d^3p}{(2\pi)^3} \omega_p (\hat{a}^\dagger_p \hat{a}_p + \frac{1}{2}[\hat{a}_p, \hat{a}^\dagger_p]) \quad (108)$$

$$= \int \frac{d^3p}{(2\pi)^3} \omega_p (\hat{a}^\dagger_p \hat{a}_p + \frac{1}{2}\delta_{pp} \cdot \mathbb{I}) \quad (109)$$

$$\cong \int \frac{d^3p}{(2\pi)^3} \omega_p \hat{a}^\dagger_p \hat{a}_p \quad (110)$$

Where the infinite absolute energy shift is tossed to get the last line, since we only measure energy differences, and $\hat{H}_{KG}$ is diagonalized!

5 Lecture 5: Scalar Quantum Field Theory

By analogy to the classical Klein-Gordon equation and Hamiltonian, a model for the (equal time, $t = 0$) quantum Klein-Gordon Hamiltonian was constructed and diagonalized, via (continuous) Fourier transform and ladder operators,

$$\hat{H}_{KG} = \frac{1}{2} \int d^3x \; \hat{\pi}^2(x) + (\nabla\hat{\phi}(x))^2 + m^2\hat{\phi}^2(x) \tag{111}$$

$$\hat{H}_{KG} = \int \frac{d^3p}{(2\pi)^3} \; \omega_p \hat{a}_p^\dagger \hat{a}_p \tag{112}$$

Where the zeroth, time, component of the momentum 4-vector $\mathbf{p}_0 = \omega_p = \sqrt{|p|^2 + m^2}$ depends on the spatial 3-vector p and the constant m^2.

This yields a representation of a one parameter subgroup of the Poincaré group, namely $U((t, 0, 0, 0)) = e^{-it\hat{H}_{KG}}$, but a true relativistic quantum field theory requires the full (projective) unitary representation of the Poincaré group, including generators for all possible transformation: 10 Lorentz + 4 translation = 14 total transformations in the Poincaré group.

To quantize, put hats on the conserved charges identified by Noether's theorem: $Q_\alpha \to \hat{Q}_\alpha$. First, consider the generators of spatial translations, namely momentum. Recall that the classical conserved current T^{0j} gives these, which is quantized: $p^j \to \hat{p}^j$.

Recall the classical energy-momentum tensor for the Klein-Gordon field

$$T^\mu{}_\nu|_{KG} = \partial^\mu\phi\partial_\nu\phi - \delta^\mu{}_\nu(\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2) \tag{113}$$

From this, quantize and calculate the conserved charge for temporal translations, namely the Hamiltonian, and conserved charges for spatial translations, namely linear momentum.

$$\hat{H}_{KG} = \int d^3x \; \hat{T}^{00} = \int \frac{d^3p}{(2\pi)^3} \omega_p \hat{a}_p^\dagger \hat{a}_p \tag{114}$$

$$\hat{p}^j = \int d^3x \; \hat{T}^{0j} = \int d^3x \; \dot{\hat{\phi}}\partial_j\hat{\phi} = \int d^3x \; \hat{\pi}\partial_j\hat{\phi}\hat{p}^j = \int \frac{d^3p}{(2\pi)^3} \; p^j \hat{a}_p^\dagger \hat{a}_p \tag{115}$$

Note that there are several choices for the ordering of $\hat{\pi}$ and $\hat{\phi}$ in the expression of $\hat{p}^j$ matters, and here is written the one that works.

Now check that the four-vector obeys the commuation relations, using the diagonalized momenta $\hat{p}^j = \int \frac{d^3p}{(2\pi)^3} \; p^j \hat{a}_p^\dagger \hat{a}_p$

$$\{Q_\alpha, Q_\beta\}_{PB} = f_{\alpha\beta}{}^\gamma Q_\gamma \to [\hat{Q}_\alpha, \hat{Q}_\beta] = if_{\alpha\beta}{}^\gamma Q_\gamma \tag{116}$$

$$\{p^\mu, p^\nu\}_{PB} = 0 \to [\hat{p}^\mu, \hat{p}^\nu] = 0 \tag{117}$$

This confirms a projective unitary representation of the *translation subgroup* of the Poincaré group, and now construct the explicit Hilbert space as a Fock space, since the operators are quantized and diagonalized via ladder operators.

To construct a Fock space, begin by defining the vacuum state, highest weight vector in the language of representation theory, $|\Omega\rangle$ such that the annihilation operator will completely obliterate it: $\hat{a}_{\mathbf{p}}|\Omega\rangle = 0, \ \forall p$, where $\mathbf{p} = (\omega_p, p)$.

The Hilbert space would then be generated via all finite linear combinations of vectors of the form $|\mathbf{p_1 p_2} \ldots \mathbf{p}_n\rangle = \hat{a}_{\mathbf{p_1}}^\dagger \hat{a}_{\mathbf{p_2}}^\dagger \ldots \hat{a}_{\mathbf{p}_n}^\dagger |\Omega\rangle$, but there is a technical issue of the n-dimensional momentum state vectors actually being improper vectors that are not normalizable, such that the scalar product needed to finish the defintion of the Hilbert space will always blow up to infinity, since $\langle p|q\rangle = (2\pi)^3 \delta^{(3)}(p-q)$. These states are also not preparable by experiment, since the state vector $|\mathbf{p_1 \, p_2} \ldots \mathbf{p}_n\rangle$ represents n delta functions in position-momentum space.

To create a normalizable state that can be used to define the Hilbert space, "smear out" the momentum states by defining a smooth (L^2) function ψ, which must be Lorentz invariant, though the invariance it is not obvious

$$|\psi\rangle = \int \frac{d^3p}{(2\pi)^3}\psi(\mathbf{p})|\mathbf{p}\rangle = \int \frac{d^3p}{(2\pi)^3}\psi(\mathbf{p})\hat{a}_p^\dagger|\Omega\rangle. \tag{118}$$

Now, introduce a method to normalize these improper vectors to a new set of improper vectors that are manifestly, more obviously, Lorentz invariant, and offer a nice parameterization to make many calculations easier.

Consider the projection operator onto a single particle state, and note that the integrand and the integral (volume element) are both separately not invariant

$$\mathbb{I}_{single} = \int \frac{d^3p}{(2\pi)^3}|p\rangle\langle p|. \tag{119}$$

Enter a reference frame where this state is invariant by multiplying by one

$$\mathbb{I}_{single} = \int \frac{d^3p}{(2\pi)^3 X(\mathbf{p})}X(\mathbf{p})|p\rangle\langle p|. \tag{120}$$

Where $X(\mathbf{p})$ is a mystery factor to make the integral and integrand invariant.

Claim: $X(\mathbf{p}) = \frac{2\omega_p}{(2\pi)^3}$, where p here must be the momentum 4-vector, since we are using the zeroth, or time, component $p_0 = \omega_p = \sqrt{|p|^2 + m^2}$.

Proof:

First, observe that $\int d^3p$ is not Poincaré invariant, but $\int d^4p$ is, such that $\int d^4p = \int d^4p'$, where $p'^\mu = \Lambda^\mu_{\ \nu}p^\nu + a^\mu$ is a Poincaré transformation, and

$\Lambda^{\mu}_{\ \nu}$ is the Jacobian of the transformation, and $det(\Lambda) = \pm 1$, $\forall \, \Lambda$ unitary transformation.

Now notice that $p^{\mu}p_{\mu} = const. = m^2$ (4-vector length invariant), whose solution is the dispersion relation for a single relativistic particle, and has two branches $\mathbf{p}_0 = \omega_p = \pm\sqrt{|p|^2 + m^2}$, where $|p|$ is the norm of the momentum (spatial) 3-vector.

Restrict to the positive upper branch, and consider the Poincaré invariant quantity

$$\int d^4 p \ \delta(\mathbf{p}_0^2 - |p|^2 - m^2)|_{\mathbf{p}_0 > 0} = \int \frac{d^3 p}{2\mathbf{p}_0}|_{\mathbf{p}_0 = \omega_p}. \tag{121}$$

Therefore, to make the single particle state projection operator from above Poncaré invariant, compare terms in the line above to the "mystery factor" expression, proving that $X(\mathbf{p}) = \frac{2\omega_p}{(2\pi)^3}$.

Thus, the "delta normalization" of 3-vectors is defined via

$$2\omega_p \delta^{(3)}(p - q). \tag{122}$$

And the renormalized, Lorentz invariant momentum 4-vector is built as

$$|\mathbf{p}\rangle = \sqrt{2\omega_p} |p\rangle = \sqrt{2\omega_p} \hat{a}_p^{\dagger} |\Omega\rangle . \tag{123}$$

And the Lorentz invariant four-length comes out to be

$$\langle \mathbf{p} | \mathbf{q} \rangle = 2(2\pi)^3 \omega_p \delta^{(3)}(p - q). \tag{124}$$

Now, to express the operators in terms of the Fock vector space we build on top of the vectors $|\mathbf{p}\rangle$, and determine the action of the generator of spacetime translations, the 4-momentum operator, $\hat{p}^{\mu}$ on the Hilbert space of momentum states (improper vectors) $|\mathbf{p_1 \, p_2} \cdots \mathbf{p_n}\rangle$.

This requires some commutation relations with the ladder operator $\hat{a}_{\mathbf{p}}$ in the following lemma.

Lemma: $[\hat{H}_{KG}, \hat{a}_p] = -\omega_p \hat{a}_p$ and $[\hat{p}^j, \hat{a}_{\mathbf{p}}] = p^j \hat{a}_p$.

Next follows the corollary, demonstrating that the operator $\hat{p}^{\mu}$ is *diagonalized* in this Hilbert space basis, such that the 4-momentum operator annihilates the vacuum state: $\hat{p}^{\mu} |\Omega\rangle = 0$.

Corollary: $\hat{p}^{\mu} |\mathbf{p_1 p_2} \cdots \mathbf{p_n}\rangle = (\sum_{j=1}^{n} p_j^{\mu}) |\mathbf{p_1 p_2} \cdots \mathbf{p_n}\rangle$.

Lorentz Invariance in the Heisenberg Picture

So, this operator allows unitary quantum spacetime translations, in the Schrödinger picture, via the exponentiated Hermitian operator quantity $U(a) = e^{-ia_\mu \hat{p}^\mu}$.

Now, to manifest any symmetries that may have not been shown in the Schröger picture, explore Lorentz invariance in the Heisenberg picture, which is also later helpful in perturbation theory. Real space calculations, at a specific spacetime location (e.g., $(t, \mathbf{x})$) are also much easier in the Heisenberg picture than in the "spread-out" Fourier transformed Schrödinger picture.

To enter the Heisenberg picture, where time is explicitly included, an operator $\mathcal{O}$ is unitarily transformed, and its time evolution is determined via the Hamiltonian in the Heisenberg equation of motion

$$\mathcal{O}_H = e^{i\hat{H}t} \mathcal{O} e^{-i\hat{H}t} \tag{125}$$

$$\frac{d\mathcal{O}_H}{dt} = i[\hat{H}, \mathcal{O}_H]. \tag{126}$$

In the Heisenberg picture, the commutation relations for the canonical position and momentum operators become

$$[\hat{\phi}_H(t,x), \hat{\phi}_H(t,y)] = [\hat{\pi}_H(t,x), \hat{\pi}_H(t,y)] = 0 \tag{127}$$

$$[\hat{\phi}_H(t,x), \hat{\pi}_H(t,y)] = i\delta^{(3)}(x-y). \tag{128}$$

Evolve the canonical position and momentum operators in time via the (spatially localized) Heisenberg equation of motion

$$\frac{d\hat{\phi}(t,x)}{dt} = i[\hat{H}_{KG}, \hat{\phi}(t,x)] = \hat{\pi}(t,x) \tag{129}$$

$$\frac{d\hat{\pi}(t,x)}{dt} = i[\hat{H}_{KG}, \hat{\pi}(t,x)] = \nabla^2 \hat{\phi}(t,x) + m^2 \hat{\phi}(t,x). \tag{130}$$

Where the second equality is gotten by using integration-by-parts. Substitute the first equality for $\hat{\pi}(t,x)$ into the second equality, and combine the second derivatives of space and time, to show that the canonical field position operator obeys the Klein-Gordon equation

$$(\partial^\mu \partial_\mu + m^2)\hat{\phi}(t,x) = 0. \tag{131}$$

This completes the development of the unitary representation of spacetime translations. Rotations and boosts are yet to be integrated into the unitary representation of the Poincaré group.

6 Lecture 6: Causality in Scalar QFT

Thus far, we have diagonalized the Klein-Gordon Hamiltonian

$$\hat{H}_{KG} = \frac{1}{2} \int d^3x \left(\hat{\pi}^2(x) + (\nabla\hat{\phi}(x))^2 + m^2\hat{\phi}^2(x) \right) = \int \frac{d^3p}{(2\pi)^3} \hat{a}_p^\dagger \hat{a}_p \qquad (132)$$

And used this to construct the unitary operator that allows us to study the dynamics of Klein-Gordon's solution to Schrödinger's equation in the Heisenberg picture

$$\hat{U} = e^{-i\hat{H}_{KG}t}. \qquad (133)$$

We found the spatial solutions $\hat{\phi}(x)$, in the Schrödinger picture, for the Klein-Gordon equation, to be the position field operators

$$\hat{\phi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(\hat{a}_p e^{ip\cdot x} + \hat{a}_p^\dagger e^{-ip\cdot x} \right) \qquad (134)$$

Notation: The spatial 3-vectors of position x and momentum p are no longer bold-faced, and the spacetime 4-vectors will be bold-faced, such that

$$\mathbf{x} = (x^0, x) \qquad\qquad = (t, x) = (x^0, x^1, x^2, x^3) \qquad (135)$$
$$\mathbf{p} = (p_0, p) \qquad\qquad = (\omega_p, p) = (x^0, x^1, x^2, x^3). \qquad (136)$$

These solutions are represented in the Heisenberg picture via

$$\hat{\phi}(t, x) = \hat{U}^\dagger \hat{\phi}(x) \hat{U} = e^{i\hat{H}_{KG}t} \hat{\phi}(x) e^{-i\hat{H}_{KG}t}. \qquad (137)$$

We still need to complete the (projective) unitary representation of the Poincaré group, including translations, boosts, and rotations, since we are doing *relativistic* quantum field theory.

The next step here is to check that $\hat{\phi}(t, x)$ respects causality, such that if two spacetime events are space-like separated, then they have no influence on each other. Note that if the two spacetime events are time-like, there may be influence.

Recall the commutation relation of the Hamiltonian (dropping subscript "KG") and the ladder operator

$$[\hat{H}, \hat{a}_p] = -\omega_p \hat{a}_p \implies e^{i\hat{H}t} \hat{a}_p e^{-i\hat{H}t} = e^{-i\omega_p t} \hat{a}_p. \qquad (138)$$

Substituting $\hat{\phi}(x)$ into the Heisenberg picture, and using the above commutation relation, we have the field oeprator in the Heisenberg picture

$$\hat{\phi}(t, x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(\hat{a}_p e^{i\mathbf{p}\cdot\mathbf{x}} + \hat{a}_p^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}} \right) \qquad (139)$$

Consider the delocalized ("smeared out") field operator $\hat{\phi}(t, x)$ as an observable that samples the field at a localized spacetime location $\mathbf{x} = (t, x)$. The question is whether this interpretation respects causality.

Consider a (projective) measurement event at (t, x) of the quantum field $\hat{\phi}(t, x)$. This disturbance of the field should fly out at the speed of light along its spacetime light cone. The fact is that measuring the field causes an instantaneous disturbance *everywhere*, but relativity is safe since we can not signal, send information, faster than the speed of light (outside of the forward light cone of the measurement event).
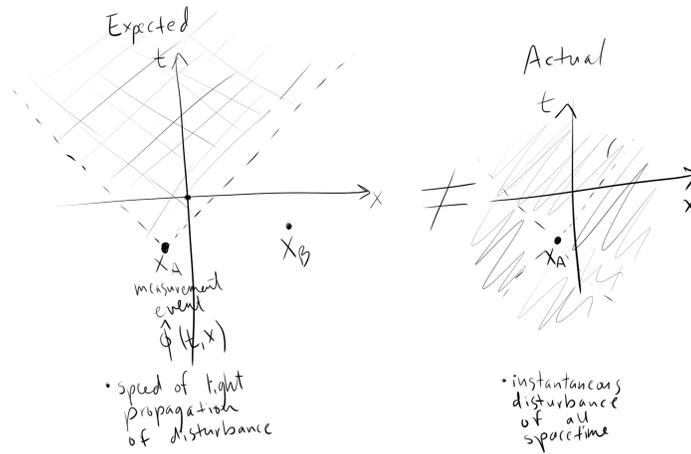
Figure 4: Sketch of expected propagation of information (at the speed of light) due to measurement event of the field $\hat{\phi}(t, x)$.

The result is that no information may be transmitted via the field across a space-like interval $((x_A - x_B)^2 < 0)$.

Now, we have to agree on which quantities are observable, and a natural guess is to study the correlation function

$$\langle 0 | \, \hat{\phi}(x^0, x) \hat{\phi}(y^0, y) \, | 0 \rangle . \tag{140}$$

This is unfortunately wrong, since the correlation function has no operational meaning, and can not be directly measured, since the field operators are not Hermitian, in general.

Digression: Interference experiment

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To study correlation functions which can be measured in the lab via experiment, consider the following interference experiment set up with a Klein-Gordon field with auxiliary modes of light used to perform measurements and generate results.
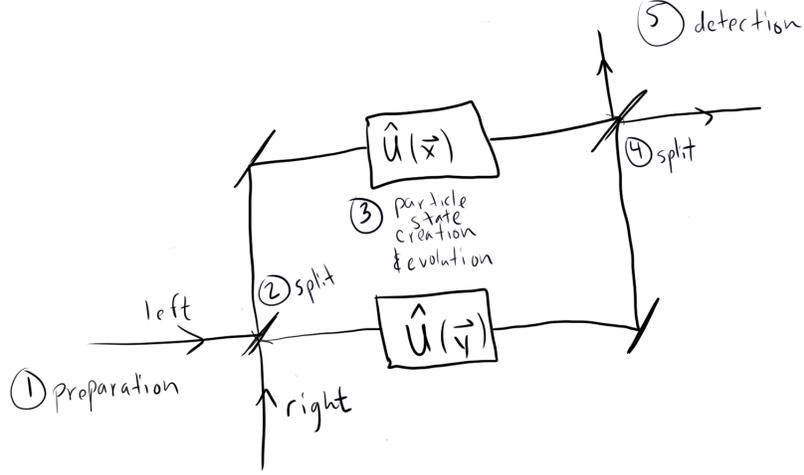

Figure 5: Schematic of experiment to measure the correlation of two spacetime locations and the Klein-Gordon field under measurement.

1. Prepare the field and auxiliary system (left and right) in the vacuum state

$$|0\rangle_{field} |0\rangle_{left} |0\rangle_{right}. \tag{141}$$

2. Apply the Hadamard gate (beam splitter) to the left and right auxiliary states

$$|0\rangle_{field} \left(\frac{1}{\sqrt{2}} \left(|0\rangle_{left} |1\rangle_{right} + |1\rangle_{left} |0\rangle_{right} \right) \right). \tag{142}$$

3. Create a particle at $\mathbf{x}$ or $\mathbf{y}$ via the unitary operator

$$\hat{U}(\mathbf{x}) \otimes |1\rangle \langle 1|_{left} \otimes \mathbb{I}_{right} + \hat{U}(\mathbf{y}) \otimes \mathbb{I}_{left} |1\rangle \langle 1|_{right} \tag{143}$$

Applied to the state in **Step 2**, which evolves to the state (dropping "left" and "right" labels)

$$\frac{1}{\sqrt{2}} \left(\hat{U}(\mathbf{y}) |0\rangle_{field} |0\,1\rangle + \hat{U}(\mathbf{x}) |0\rangle_{field} |1\,0\rangle \right) \tag{144}$$

4. Apply a second beam splitter to the auxiliary states to check for interference in the final state

$$\frac{1}{2} \left(\hat{U}(\mathbf{x}) + \hat{U}(\mathbf{y}) \right) |0\rangle_{field} |0\,1\rangle + \frac{1}{2} \left(\hat{U}(\mathbf{x}) - \hat{U}(\mathbf{y}) \right) |0\rangle_{field} |1\,0\rangle \tag{145}$$

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5. Detect auxiliary states $|0\,1\rangle$ and $|1\,0\rangle$.

The probability of measuring one of the states, $|0\,1\rangle$, for example, is

$$\mathcal{P}(|0\,1\rangle) = \frac{1}{4} \langle 0|_{field} \left(\hat{U}^\dagger(\mathbf{x}) + \hat{U}^\dagger(\mathbf{y}) \right) \left(\hat{U}(\mathbf{x}) + \hat{U}(\mathbf{y}) \right) |0\rangle_{field} \tag{146}$$

$$= \frac{1}{2} + \frac{1}{2} Re \left[\langle 0|_{field} \, \hat{U}^\dagger(\mathbf{x}) \hat{U}(\mathbf{y}) \, |0\rangle_{field} \right] \tag{147}$$

$$= \frac{1}{2} + \frac{1}{2} Re \left[\langle 0|_{field} \, e^{-i\epsilon\hat{\phi}(\mathbf{x})} e^{i\epsilon\hat{\phi}(\mathbf{y})} \, |0\rangle_{field} \right] \tag{148}$$

$$\mathcal{P}(|0\,1\rangle) = \frac{1}{2} + \frac{1}{2} Re \left[\langle 0|_{field} \, e^{-i\epsilon\hat{\phi}(\mathbf{x})+i\epsilon\hat{\phi}(\mathbf{y})+\frac{1}{2}\epsilon^2[\hat{\phi}(\mathbf{x}),\hat{\phi}(\mathbf{y})]} \, |0\rangle_{field} \right] \tag{149}$$

$$\tag{150}$$

Where the last line is gotten via the Baker-Campbell-Hausdorff relation, since the two fields operators do not necessarily commute, and the interference between the two measurement events is determined by whether the commutator is zero or nonzero, which is calculated

$$[\hat{\phi}(\mathbf{x}), \hat{\phi}(\mathbf{y})] = \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{4\omega_p\omega_q}} \left([\hat{a}_p, \hat{a}_q^\dagger] e^{-i\mathbf{p}\cdot\mathbf{x}+i\mathbf{q}\cdot\mathbf{y}} + [\hat{a}_p^\dagger, \hat{a}_q] e^{i\mathbf{p}\cdot\mathbf{x}-i\mathbf{q}\cdot\mathbf{y}} \right) \tag{151}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} - e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \right) \cdot \mathbb{I} \tag{152}$$

$$[\hat{\phi}(\mathbf{x}), \hat{\phi}(\mathbf{y})] = \Delta(\mathbf{x} - \mathbf{y}) \cdot \mathbb{I} \tag{153}$$

Where the commutation relation $[\hat{a}_p, \hat{a}_q^\dagger] = (2\pi)^3 \delta^{(3)}(p - q) \cdot \mathbb{I}$ is used, and the quantity $\Delta(\mathbf{x}-\mathbf{y})$, the **correlation function**, which is Lorentz invariant, must be zero when $\mathbf{x}$ and $\mathbf{y}$ are space-like separated, such that $(\mathbf{x} - \mathbf{y})^2 < 0$.

Consider the space-like separation, $(\mathbf{x} - \mathbf{y})^2 < 0$, and enter a reference frame where $\mathbf{x} - \mathbf{y} = (0, x - y)$, and the correlation function $\Delta(\mathbf{x} - \mathbf{y})$ becomes zero!

$$\Delta(\mathbf{x} - \mathbf{y}) = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{|p|^2 + m^2}} \left(e^{ip\cdot(x-y)} - e^{-ip\cdot(x-y)} \right) \tag{154}$$

$$= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{|p|^2 + m^2}} \left(e^{ip\cdot(x-y)} - e^{ip\cdot(x-y)} \right) \tag{155}$$

$$\Delta(\mathbf{x} - \mathbf{y}) = 0 \tag{156}$$

Where the second line is gotten by applying the Lorentz transformation $(x - y) \to -(x - y)$, which is allowed for in a space-like interval.
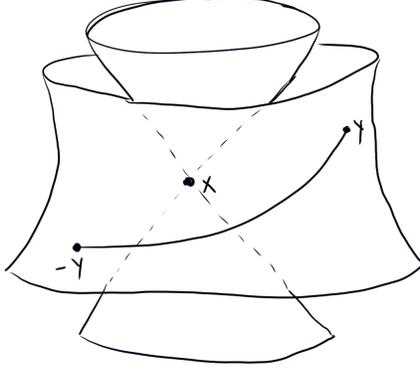
Figure 6: Sketch of spatial inversion of spacetime location $\mathbf{y}$.

Therefore, information does not travel faster than the speed of light, and the field operators respect causality when they are space-like separated! Note that this does not yet prove that the entire theory is causal.

As a side note, consider the case where the two field measurements are time-like separated, such that $(\mathbf{x} - \mathbf{y})^2 > 0$, and enter a reference frame where $\mathbf{x} - \mathbf{y} = (t, 0, 0, 0)$. In this reference frame of time-like separation, the correlation function can **not** be zero

$$\Delta(\mathbf{x} - \mathbf{y}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(e^{-i\omega_p t} - e^{i\omega_p t} \right) \tag{157}$$

$$= \frac{1}{4\pi^2} \int_m^\infty dE \sqrt{E^2 + m^2} e^{-iEt} \tag{158}$$

$$\Delta(\mathbf{x} - \mathbf{y}) \sim e^{-imt} - e^{imt} \neq 0. \tag{159}$$

Return to consider the (non-physical) two-point correlation function in a space-like interval

$$D(\mathbf{x} - \mathbf{y}) = \langle 0 | \hat{\phi}(\mathbf{x}) \hat{\phi}(\mathbf{y}) | 0 \rangle \tag{160}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} e^{-i\mathbf{p} \cdot (\mathbf{x} - \mathbf{y})} \tag{161}$$

$$= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{|p|^2 + m^2}} e^{ip \cdot (x-y)} \tag{162}$$

$$= \frac{m}{4\pi^2} \frac{1}{|x - y|} K_1(m|x - y|) \tag{163}$$

$$D(\mathbf{x} - \mathbf{y}) \sim e^{-m|x-y|} \neq 0 \tag{164}$$

Where $K_1(x)$ denotes the Hankel function.

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So, when $\mathbf{x}$ and $\mathbf{y}$ are space-like separated, this correlation function can not be zero, and can therefore not carry any information, as it would have to travel faster than the speed of light.

An important application of the correlation function is the **Feynman propagator**, which is used later in perturbative expansions for describing interactions.

$$\Delta_F(\mathbf{x} - \mathbf{y}) = \begin{cases} D(\mathbf{x} - \mathbf{y}), & x^0 > y^0 \\ D(\mathbf{y} - \mathbf{x}), & x^0 < y^0 \end{cases} \tag{165}$$

$$\Delta_F(\mathbf{x} - \mathbf{y}) = \langle 0 | \mathcal{T}[\hat{\phi}(\mathbf{x})\hat{\phi}(\mathbf{y})] | 0 \rangle \tag{166}$$

Where $\mathcal{T}[\ ]$ is called the **time-ordering operator**

$$\mathcal{T}[\hat{\phi}(\mathbf{x})\hat{\phi}(\mathbf{y})] = \begin{cases} \hat{\phi}(\mathbf{x})\hat{\phi}(\mathbf{y}), & x^0 > y^0 \\ \hat{\phi}(\mathbf{y})\hat{\phi}(\mathbf{x}), & x^0 < y^0 \end{cases} \tag{167}$$

Another important definition of the Feynman propagator is in terms of complex variables and contour integrals.

Lemma:

$$\Delta_F(\mathbf{x} - \mathbf{y}) = \int \frac{d^4 p}{(2\pi)^4} \frac{ie^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}}{|\mathbf{p}|^2 - m^2 + i\epsilon} \tag{168}$$

$$= \int \frac{d^3 p}{(2\pi)^3} \int_C \frac{dp^0}{2\pi} \frac{ie^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}}{|\mathbf{p}|^2 - m^2 + i\epsilon} \tag{169}$$

Where the integrand of the contour integral over the complex variable p^0 has two poles at $\pm i\epsilon$, and the contour is taken along the real p^0 axis, and closed in the upper or lower half-plane, depending on the value of p^0.
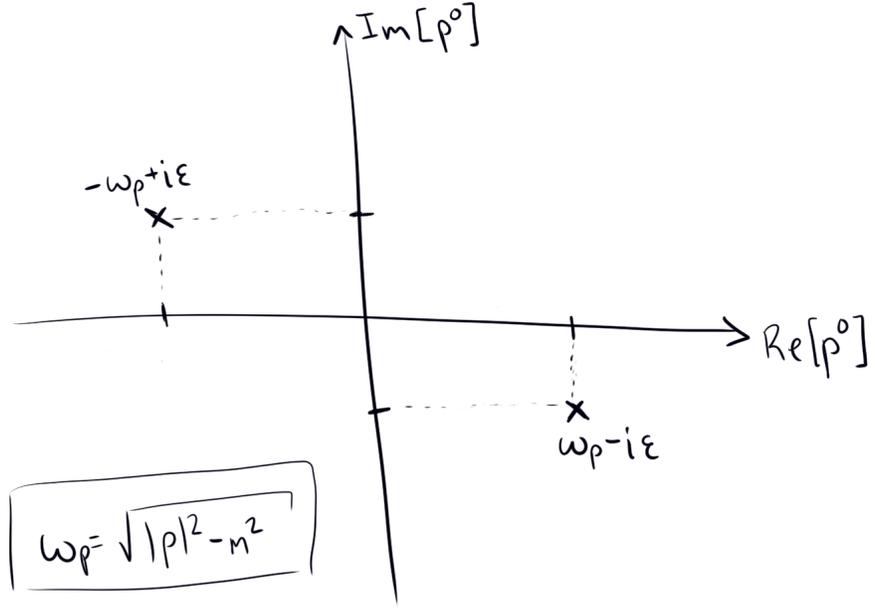
Figure 7: Poles of the Feynman propagator, where the contour may be taken in the upper half-plane for $t < 0$, and in the lower half-plane for $t > 0$, where $t = x^0 - y_0$.

Lastly, an observation that the Feynman propagator, which is related thus far to the two-point correlation function, is also a Green's function (inverse of a differential operator) for the Klein-Gordon partial differential equation

$$(\partial_0^2 - \nabla^2 + m^2)\Delta_F(\mathbf{x} - \mathbf{y}) = -i\delta^{(4)}(\mathbf{x} - \mathbf{y}) \tag{170}$$

$$\sim \hat{\mathbb{L}} \cdot \hat{\mathbb{L}}^{-1} = \mathbb{I} \tag{171}$$

Where the left-hand side is the product of a linear differential operator, the Klein-Gordon operator, and its inverse, the Feynman propagator, and the right-hand side of the equation is, in essence, the identity.

7 Lecture 7: Representing Symmetries in QFT

Here we finish study of the quantum Klein-Gordon field by working out how the Lorentz group is unitarily represented on the space of states on the Klein-Gordon field. Recall that for a relativistic quantum field theory, we must have a (projective) unitary representation of the Poincaré group $U(\Lambda, a)$ on a Hilbert space $\mathcal{H}$. Thus far, we have constructed the Hilbert space, with respect to some norm $|| \cdot ||$, as the space of states gotten by applying creation operators to the vacuum state

$$\mathcal{H} = span\{\hat{a}^\dagger_{\mathbf{p}_1} \hat{a}^\dagger_{\mathbf{p}_2} \ldots \hat{a}^\dagger_{\mathbf{p}_n} |\Omega\rangle\}^{||\cdot||} \tag{172}$$

Digression: Continuous Groups of Symmetries

The central idea of a group with a continuous manifold structure (e.g., Lie groups) is to study symmetries close to, localized to, the identity and then exponentiate to larger, more global, elements. Important continuous operations on this manifold $\mathcal{M}$ include (closed) composition and inverse.
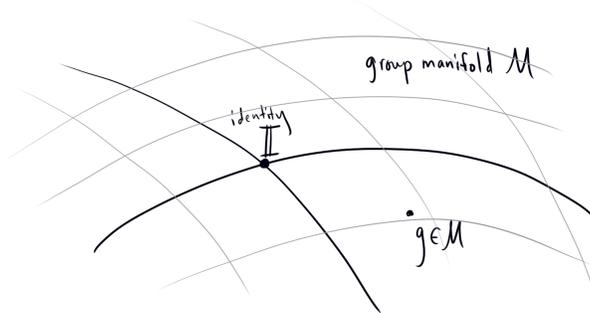

Figure 8: Sketch of a group manifold with identity $\mathbb{I}$ and an element of the manifold g.

$$\text{Closure:} \quad \mathcal{M} \times \mathcal{M} \to \mathcal{M} \tag{173}$$
$$g \times h \to g \cdot h \tag{174}$$
$$\text{Inverse:} \quad \mathcal{M} \to \mathcal{M} \tag{175}$$
$$g \to g^{-1} \tag{176}$$

The Poincaré group is an example of a continuous Lie group, and to understand its structure, consider elements $g \in \mathcal{M}$ infinitesimally close to the identity $\mathbb{I}$, such that $g - \mathbb{I} \sim \mathcal{O}(\epsilon)$. These infinitesimal elements are elements of the tangent space $T_{\mathbb{I}} \mathcal{M}$, which is a linear space and is called a *Lie algebra*. The basis vectors of $T_{\mathbb{I}} \mathcal{M}$ are written as x^j, $j = 1, 2, \ldots, dim(\mathcal{M})$.

Examples of Tangent Spaces

1. 1×1 unitary matrices $U(1) = \{\phi \in \mathbb{C} : |\phi|^2 = 1\}$
   $$\to T_1 U(1) = \{z : (1 + \epsilon z)^*(1 + \epsilon z) = 1 \text{ to order } \epsilon, \text{ s.t. } Re[z] = 0)\}$$

2. 3×3 Euclidean rotation matrices $O(3) = \{O \in \mathcal{M}_3(\mathbb{R}) : O^T O = \mathbb{I}\}$
   $$\to T_{\mathbb{I}} O(3) = \{X : (\mathbb{I} + \epsilon X)^T(\mathbb{I} + \epsilon X) = \mathbb{I} \text{ to order } \epsilon, \text{ s.t. } X + X^T = 0\}$$
   - Note that including inversions promotes $O(3)$ to $SO(3)$.
   - Basis of $O(3) =$
   $$\{X = \sum_{j=1}^3 X_j J^j : J^1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, J^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, J^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \}$$

3. 4×4 Lorentz transformation matrices $G = \{\Lambda : \eta_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma = \eta_{\rho\sigma}\}$

   $$\to T_{\mathbb{I}} G = \{\omega : \eta_{\mu\nu}(\mathbb{I} + \epsilon\omega)^\mu{}_\rho(\mathbb{I} + \epsilon\omega)^\nu{}_\sigma = \eta_{\rho\sigma}\}$$
   $$= \{\omega : \text{antisymmetric, s.t. } \omega^{\mu\nu} = -\omega^{\nu\mu}\}$$

   Note the "upstairs" covariant indices on ω

   Using "downstairs" contravariant indices

   requires multiplication by the metric $\eta_{\mu\nu}$ as above.

   $$= \{\omega : \text{change of basis to } \omega = \sum_{j=1}^6 \frac{1}{2}\Omega_j J^j\}$$

   Change indices via the bijection $j \to (\rho\,\sigma)$, with $0 < \rho < \sigma \leq 3$
   $$1 \to (0\,1); \ 2 \to (0\,2); \ 3 \to (0\,3)$$
   $$4 \to (1\,2); \ 5 \to (1\,3); \ 6 \to (2\,3)$$

   $$= \{\omega : \text{with index bijection } \omega = \sum_{(\rho\sigma)} \frac{1}{2}\Omega_{(\rho\sigma)}J^{(\rho\sigma)}\} \quad (\textbf{Exercise})$$

   This bijection allows the basis to be expressed

   by the formula $[J^{(\rho\sigma)}]^\mu{}_\nu = \eta^{\rho\sigma}\delta^\sigma{}_\nu - \eta^{\sigma\mu}\delta^\rho{}_\nu$

Understanding the structure of a linear space infinitesimally close to the identity yields information about the whole Lie group and the global structure of the manifold. Also note that multiplication on this linear space is a continuous map that strongly determines the group structure on the manifold.

To demonstrate how a Lie algebra on the (linear) tangent space produces a Lie group, and vice versa, consider the element of the tangent space of a manifold $X \in T_{\mathbb{I}}\mathcal{M}$, such that $\mathbb{I} + sX \sim e^{\epsilon X} \in \mathcal{M}$, to order ϵ.

Define the function $g(s) = \lim_{n \to \infty}(\mathbb{I} + \frac{s}{n}X)^n = e^{sX} \in \mathcal{M}$. Therefore, every element of the Lie algebra (tangent space to the identity) determines an element

of the Lie group (manifold). To go the other way, and determine the Lie algebra from the Lie group, apply the logarithm map.
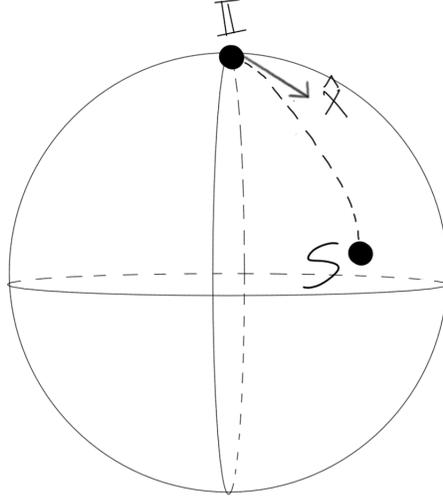

Figure 9: Figure of exponentiation map on Lie group (e.g., $O(3)$) manifold. Translation from identity a distance s along the X direction. Resulting position on manifold is called the exponential of sX.

Algebraic structure on tangent space

The algebraic structure on the tangent space is defined by the *group commutator* on the manifold, which is also a group.

$$[\,,\,] : \mathcal{M} \times \mathcal{M} \to \mathcal{M} \tag{177}$$

$$(g, h) \to [g, h] = ghg^{-1}h^{-1}, \, \forall \, g, h \in \mathcal{M} \tag{178}$$

The group commutator also pushes forward to a mapping of the tangent space

$$[\,,\,] : T_{\mathbb{I}}\mathcal{M} \times T_{\mathbb{I}}\mathcal{M} \to T_{\mathbb{I}}\mathcal{M} \tag{179}$$

Consider the following group commutator which is an element of the manifold

$$[\mathbb{I} + \epsilon X, \mathbb{I} + \delta Y] = (\mathbb{I} + \epsilon X)(\mathbb{I} + \delta Y)(\mathbb{I} - \epsilon X)(\mathbb{I} - \delta Y) \tag{180}$$

$$= \mathbb{I} + \epsilon \delta (XY - YX) + \mathcal{O}(\epsilon^2) + \mathcal{O}(\delta^2) + \dots \tag{181}$$

$$\sim \mathbb{I} + \epsilon \delta [X, Y] \tag{182}$$

Where ϵ and δ are small, independent parameters, and the Lie algebra commutator $[X, Y]$ is, therefore, an element of the tangent space of the manifold, such that $[X, Y] \in T_{\mathbb{I}}\mathcal{M}$.

Examples of Lie algebra commutators

1. $U(1)$: trivial

2. $O(3)$: $[J^j, J^k] = -\epsilon^{jk}_{\ l} J^l$

3. Lorentz group: $[J^{\rho\sigma}, J^{\tau\nu}] = \eta^{\sigma\tau} J^{\rho\nu} - \eta^{\rho\tau} J^{\sigma\nu} + \eta^{\rho\nu} J^{\sigma\tau} - \eta^{\sigma\nu} J^{\rho\tau}$

    E.g.,

$$[J^{01}]^\mu_{\ \nu} = \left(\begin{array}{cc|cc} 0 & 1 & & \\ 1 & 0 & \mathbf{0} & \\ \hline & & 0 & 0 \\ \mathbf{0} & & 0 & 0 \end{array}\right), \quad [J^{12}]^\mu_{\ \nu} = \left(\begin{array}{cc|cc} & & 0 & 0 \\ \mathbf{0} & & -1 & 0 \\ \hline 0 & 1 & & \\ 0 & 0 & \mathbf{0} & \end{array}\right)$$

$$[J^{01}, J^{12}] = -J^{13} = \left(\begin{array}{cc|cc} & & 0 & 0 \\ \mathbf{0} & & 0 & -1 \\ \hline 0 & 0 & & \\ 0 & 1 & \mathbf{0} & \end{array}\right)$$

    Special names for this particular Lie group elements in the Lorentz transformation

    Generators of boosts (pure boost by exponentiating, s.t. $e^{\frac{1}{2}\epsilon_j K^j}$):
    $$J^{0j} = K^j$$

    Generators of rotations (elements of $O(3) \subset$ Lorentz group):
    $$J^{12} = J^1, \ J^{13} = J^2, \ J^{23} = J^3$$

Representations of Lie groups

Consider a representation π of the Lie algebra $T_{\mathbb{I}}\mathcal{M}$ which is a linear map from the tangent space to the Hilbert space of bounded linear operators, such that the Lie bracket property is preserved, such that $[\pi(X), \pi(Y)] = \pi([X, Y])$

$$\pi : T_{\mathbb{I}}\mathcal{M} \to \mathcal{B}(\mathcal{H}) \tag{183}$$

This representation is exponentiated to a representation of the Lie group manifold $\mathcal{M}$

$$\pi(g = e^{\epsilon X}) = e^{\epsilon \pi(X)} \tag{184}$$

This shows that one can either try to find matrices that obey the Lie group law, or, more easily, focus on the Lie algebra (linear space) and find matrices that obey the Lie bracket property; Lie group representations are often not worked with directly, but the elements of the Lie algebra can just be exponentiated to obtain the representation of the Lie group.

To implement a general Lorentz transformation on the Hilbert space of states allowed in the Klein-Gordon field, Noether's theorem, as well as the *inverse*

Noether's theorem, is employed. Noether's theorem allows conserved currents to be derived from symmetry transformations. Information in thrown out when integrating the currents over space $\int d^3x$ to get the conserved charge, but the time-like component is left alone, and information about the structure of the symmetry transformation is conserved, which can be gotten back by the inverse Noether's theorem.

Recap of Noether's theorem: For each symmetry of a field, with respect to a coordinate transformation $x \rightarrow \exp^{\epsilon J^{\rho\sigma}}$, there exists a conserved current per $J^{\rho\sigma}$. For example, the group of Lorentz transformations is described by six independent parameters, the generators of the transformation, associated with six conserved currents. The conserved currents from the symmetries of the Lorentz transformation have the form

$$\mathcal{J}^{(\rho\sigma)} = \mathbf{x}^\rho T^{\mu\sigma} - \mathbf{x}^\sigma T^{\mu\rho} \tag{185}$$

Where $\mathbf{x}^\mu$ are 4-vector spacetime coordinates, and $T^{\mu\nu}$ are elements of the energy-momentum tensor. The conserved charges are then gotten by integrating over space

$$Q^{(\rho\sigma)} = \int d^3x\, \mathcal{J}^{(\rho\sigma)} = \int d^3x\, (\mathbf{x}^\rho T^{0\sigma} - \mathbf{x}^\sigma T^{0\rho}) \tag{186}$$

"Noether's theorem is really just a fancy telescoping series in disguise."

The arguably more profound statement regarding conserved charges and symmetries is the inverse Noether's theorem.

Inverse Noether's theorem: Conserved charges are the generators, represent the Lie algebra, of the symmetry transformations from which they came, and generate canonical transformations, or representations of the symmetries, on phase space. Classically,

$$\{Q^{(\rho\sigma)}, Q^{(\tau\nu)}\}_{PB} = \eta^{\sigma\tau}Q^{\rho\nu} - \eta^{\rho\tau}Q^{\sigma\nu} + \eta^{\rho\nu}Q^{\sigma\tau} - \eta^{\sigma\nu}Q^{\rho\tau} \tag{187}$$

Propose that we "just put hats on" the conserved charges, and check that they obey the Lie algebra of the Lorentz group. It turns out that this works for free theories, and we have, at least, one representation of the Lie algebra of the Lorentz group in the context of one, the Klein-Gordon, quantum field.

$$\hat{Q}^{\mu\nu} = \int d^3x(x^\mu \hat{T}^{0\nu} - x^\nu \hat{T}^{0\mu}) \tag{188}$$

Consider the $0j^{th}$ conserved charge, using the specialized notation, and check

that $\hat{Q}^{0j} = \hat{K}^j$ does indeed generate boosts and is time independent.

$$\hat{Q}^{0j} = \hat{K}^j = \int d^3x \,(x^j \hat{T}^{00} - x^0 \hat{T}^{0j}) \tag{189}$$

$$\hat{K}^j = -t\,\hat{P}^j + \int d^3x \, x^j \,\hat{\mathscr{H}}(t,x) \tag{190}$$

$$\frac{d\hat{K}^j}{dt} = -\hat{P}^j + i[\hat{H}, \int d^3x \, x^j \,\hat{\mathscr{H}}(t,x)] \tag{191}$$

$$0 = \hat{P}^j + i[\hat{H}, \hat{K}^j] \tag{192}$$

$$[\hat{H}, \hat{K}^j] = -i\hat{P}^j. \tag{193}$$

Where $\hat{P}^j$ is the total field momentum, and $\hat{\mathscr{H}}$ is the Hamiltonian density, which does not commute with the Hamiltonian $\hat{H}$. The third line cancels the time dependence of the two right-hand side terms. This shows that an infinitesimal shift in time and an infinitesimal boost is equal to an infinitesimal shift in space.

Similarly for the generators of rotation, which are manifestly time independent, it must be checked that they obey the correct Lie algebra.

$$\hat{J}^{jk} = \int d^3x \, \hat{\pi}(x)(x^j \partial_k - x^k \partial_j)\hat{\phi}(x) \tag{194}$$

To perform a Lorentz transformation on the Hilbert state space, a unitary operator is created by putting some parameters Ω in front of the generators of rotation and exponentiating

$$\hat{U}(\Lambda) = e^{-\frac{1}{2}\Omega_{\rho\sigma}\hat{J}^{\rho\sigma}} \tag{195}$$

There are 9 commutation relations, that must be checked (**Exercises**), that yield the full Lie algebra of the Poincaré group.

$$[\hat{J}^j, \hat{J}^k] = -i\epsilon^{jk}{}_l \hat{J}^l \tag{196}$$

$$[\hat{J}^j, \hat{K}^k] = -i\epsilon^{jk}{}_l \hat{K}^l \tag{197}$$

$$[\hat{K}^j, \hat{K}^k] = i\epsilon^{jk}{}_l \hat{J}^l \tag{198}$$

$$[\hat{J}^j, \hat{P}^k] = -i\epsilon^{jk}{}_l \hat{P}^l \tag{199}$$

$$[\hat{K}^j, \hat{P}^k] = i\delta^{jk}\hat{H} \tag{200}$$

$$[\hat{K}^j, \hat{H}] = i\hat{P}^j \tag{201}$$

$$[\hat{J}^j, \hat{H}] = [\hat{P}^j, \hat{H}] = [\hat{P}^j, \hat{P}^k] = 0 \tag{202}$$

This now demonstrates how the Klein-Gordon field gives a full (Lie algebra) representation of the Poincaré group, and proves that to perform a Poincaré transformation on a state of Klein-Gordon particles, one simply applies a unitary transformation via exponentiation of the above operators, which are the generators of transformations.

8 Lecture 8: Interactions in QFT

Thus far, we have studied the Klein-Gordon quantum field, which evolves with time in the Heisenberg picture via the Klein Hamiltonian H_{KG}, the generator of time translations

$$\hat{\phi}(t,x) = e^{i\hat{H}_{KG}t}\hat{\phi}(0,x)e^{-i\hat{H}_{KG}t}. \tag{203}$$

We have obtained a full unitary representation of the Poincaré group for the Klein-Gordon field by constucting a space of states in terms of the field position operator $\hat{\phi}$ and the field momentum operator $\hat{\pi}$ via the generator of time translation $\hat{H}$, the generators of spatial translation $\hat{P}^j$, and the conserved charges $\hat{Q}^{\mu\nu}$. This is a *free theory*, where the dynamics of two or more spacetime events evolve completely independently of each other with no interactions between particles and field, and is relatively easy to solve.

To attempt to account for interactions, construct a Hilbert space spanned by states of the form $\{\hat{a}_p^\dagger \hat{a}_q^\dagger |0\rangle\}$, and add a (spatially localized) momentum distribution

$$|\Phi_2\rangle = \int \frac{d^3p\,d^3q}{(2\pi)^6}\,\phi_x(p)\phi_y(q)\cdot\hat{a}_p^\dagger\hat{a}_q^\dagger|0\rangle \tag{204}$$

This states evolves according to the Hamiltonian

$$|\Phi_2(t)\rangle = e^{-i\hat{H}_{KG}t}|\Phi_2\rangle \tag{205}$$

$$= \int \frac{d^3p\,d^3q}{(2\pi)^6}\,\phi_x(p)\phi_y(q)e^{-i\hat{H}_{KG}t}\hat{a}_p^\dagger e^{i\hat{H}_{KG}t}e^{-i\hat{H}_{KG}t}\hat{a}_q^\dagger e^{i\hat{H}_{KG}t}|0\rangle \tag{206}$$

Where $\hat{H}_{KG}$ is quadratic in the creation operators $\hat{a}_p^\dagger$, meaning that the quantity $e^{-i\hat{H}_{KG}t}\hat{a}_p^\dagger e^{i\hat{H}_{KG}t}$ is linear in the creation operators $\hat{a}_p^\dagger$. Therefore, the particles eveolve independently of each other in this attempted formalism, and there are no interactions, which is unphysical for an interacting theory.

Desired characteristics of the interactions that we are attempting to describe are

1. Model physical experiments

2. Maintain Lorentz invariance

3. Local interactions

To fulfill these characteristics, we consider studying models with (classical) Lagrangian densities of the form

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi(x))(\partial^\mu\phi(x)) - \frac{1}{2}m^2\phi^2(x) - \sum_{n\geq 3}^{\infty}\frac{\lambda_n}{n!}\phi^n(x) \tag{207}$$

It will later be shown that Lagrangian densities with $n > 4$ are irrelevant to observable physics, and $n = 3$ leads to instabilities, and neither case is renormalizable. Therefore, the only relevant *interacting scalar quantum field theory* is the $n = 4$ case

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi(x))(\partial^\mu \phi(x)) - \frac{1}{2}m^2 \phi^2(x) - \frac{\lambda}{4!}\phi^4(x) \tag{208}$$

In a quantum field theory, interactions are handled in several ways

1. Perturbation theory

    Expand Hamiltonian in Taylor series in terms of a small parameter

    Leads to a solvable model when this parameter is set to zero

    Feynman diagrams systematically handle all interactions in infinite series

2. Variational methods

    Approximates the system and minimizes error parameters

3. Monte Carlo sampling

4. Exact solutions

    Bethe Ansatz in $(1 + 1)$ dimensions

    Topological QFT in $(2 + 1)$ dimensions

    Supersymmetry in higher dimensions

    Large N limit

Perturbation theory

Consider the "small" addition $\hat{H}_{int}$ to the free theory Hamiltonian $\hat{H}_0$ to make the full Hamiltonian $\hat{H}$

$$\hat{H} = \hat{H}_0 + \hat{H}_{int} \tag{209}$$

Technically, we demand that $||\hat{H}_{int}||_\infty << 1$, but it often happens that $||\hat{H}_{int}||_\infty \to \infty$, where $||\hat{H}_{int}||_\infty$ is the largest eigenvalue that dominates the error estimates. Therefore, we pretend that $||\hat{H}_{int}||_\infty << 1$, and solve the time-dependent Schrödinger equation

$$i\frac{d}{dt}\ket{\psi} = \hat{H}\ket{\psi} \tag{210}$$

Interaction picture

Enter a new reference frame, the **interaction picture**, or the **Heisenberg picture**, where all states and operators from the Schrödinger picture, denoted by subscript "S", are transformed via

$$\ket{\psi_I(t)} = e^{i\hat{H}_0 t}\ket{\psi_S(t)} \tag{211}$$

$$\mathcal{O} = e^{i\hat{H}_0 t}\,\mathcal{O}_S\,e^{-i\hat{H}_0 t} \tag{212}$$

The time evolution on the interaction space of states is then

$$i\frac{d}{dt}\left|\psi_I(t)\right\rangle = i\frac{d}{dt}e^{i\hat{H}_0 t}\left|\psi_S(t)\right\rangle \tag{213}$$

$$= -\hat{H}_0 e^{i\hat{H}_0 t}\left|\psi_S(t)\right\rangle + i e^{i\hat{H}_0 t}\frac{d}{dt}\left|\psi_S(t)\right\rangle \tag{214}$$

$$= -\hat{H}_0 e^{i\hat{H}_0 t}\left|\psi_S(t)\right\rangle + i e^{i\hat{H}_0 t}(-i\hat{H}\left|\psi_S(t)\right\rangle) \tag{215}$$

$$= (-\hat{H}_0 e^{i\hat{H}_0 t} + e^{i\hat{H}_0 t}(\hat{H}_0 + \hat{H}_{int}))\left|\psi_S(t)\right\rangle \tag{216}$$

$$= (-\hat{H}_0 e^{i\hat{H}_0 t} + e^{i\hat{H}_0 t}(\hat{H}_0 + \hat{H}_{int}))\cdot e^{-i\hat{H}_0 t}e^{i\hat{H}_0 t}\cdot\left|\psi_S(t)\right\rangle \tag{217}$$

$$i\frac{d}{dt}\left|\psi_I(t)\right\rangle = (\hat{H}_{int})_I(t)\left|\psi_I(t)\right\rangle \tag{218}$$

Note: From here we drop the subscript "I" on the interaction Hamiltonian

$$(\hat{H}_{int})_I(t) \to \hat{H}_{int}(t). \tag{219}$$

The interacting time-dependent solution is

$$\left|\psi_I(t)\right\rangle = \hat{U}(t,t_0)\left|\psi_I(t_0)\right\rangle. \tag{220}$$

Where the operator $\hat{U}(t,t_0)$ is the propagator, and satisfies the equation

$$i\frac{d}{dt}\hat{U}(t,t_0) = \hat{H}_{int}(t)\hat{U}(t,t_0) \tag{221}$$

Integrating this equation with respect to t yields a constraint on the propagator

$$\hat{U}(t,t_0) = \mathbb{I} - i\int_{t_0}^{t}dt'\,\hat{H}_{int}(t')\hat{U}(t',t_0) \tag{222}$$

One way to solve for $\hat{U}(t,t_0)$ is to guess a solution and check if both sides of the constraint equation are equal.

Another approach is through *fixed point iteration*

1. Make a guess for $\hat{U}(t,t_0)$

2. Evaluate how wrong it is

3. Minimize error by adding and/or modifying terms to guess

4. Repeat, by substituting the old right-hand side into the new right-hand side, until the left-hand side and the right-hand side of the constraint approaach each other

The repeated substitution of the propagator into the constraint equation produces the **Dyson series**, where the nth has the form

$$\hat{U}(t,t_0) = (-i)^n\int_{t_0}^{t}dt'\int_{t_0}^{t'}dt''\cdots\int_{t_0}^{t^{(n-1)}}dt^{(n-1)}\,\hat{H}_{int}(t')\hat{H}_{int}(t'')\ldots\hat{H}_{int}(t^{(n-1)}) \tag{223}$$

By the triangle inequality and the product inequality, the norm of the nth term has an upper bound

$$|| \int \cdots ||_\infty \le \frac{(t-t_0)^n}{n!} (||\hat{H}_{int}||_\infty^*)^n. \tag{224}$$

And, since $\hat{H}_{int}(t)$ is just unitarily rotated from $\hat{H}_{int}$

$$||\hat{H}_{int}||_\infty^* = \sup_{t' \in [t,t_0]} ||\hat{H}_{int}(t')||_\infty = (||\hat{H}_{int}||_\infty)_S. \tag{225}$$

If $||\hat{H}_{int}||_\infty << 1$, the series has a nonzero radius of convergence.

Theorem: $\hat{U}(t,t_0) = \mathcal{T}[e^{-i\int_{t_0}^t dt' \, \hat{H}_{int}(t')}]$, where $\mathcal{T}[\,]$ is the time-ordering operator.

To prove, expand the right-hand side in a Taylor series, apply time-ordering, and check that the two sides are equal.

Observables in QFT

An important observable in QFT is scattering cross sections in scattering experiments. Namely, the $\mathcal{S}$-matrix is determined by Green's functions (n-point correlation functions)

$$G^{(n)}(x_1, x_2, \ldots, x_n) = \langle \Omega | \mathcal{T}[\hat{\phi}_{1H} \hat{\phi}_{2H} \ldots \hat{\phi}_{nH}] | \Omega \rangle \tag{226}$$

Where $|\Omega\rangle$ is the vacuum state of the Hamiltonian, and the subscript "H" denotes the Heisenberg picture, such that $\hat{\phi}_{jH} = \hat{\phi}(\mathbf{x}_j) = \hat{\phi}(t_j, x_j)$.

Claim:

$$G^{(n)}(x_1, x_2, \ldots, x_n) = \frac{\langle 0 | \mathcal{T}[\hat{\phi}_{1I} \hat{\phi}_{2I} \ldots \hat{\phi}_{nI} \hat{\mathcal{S}}] | 0 \rangle}{\langle 0 | \hat{\mathcal{S}} | 0 \rangle} \tag{227}$$

Where $\langle \phi | \hat{\mathcal{S}} | \psi \rangle = \lim_{t_\pm \to \pm\infty} \langle \phi | \hat{U}(t_+, t_-) | \psi \rangle$, and $\hat{H}_0 | 0 \rangle = 0$.

Proof:
Assume that $t_1 > t_2 > \cdots > t_n$.
Then the right-hand side of the numerator reads

$$\langle 0 | \hat{U}(\infty, t_1) \hat{\phi}_{1I} \hat{U}(t_1, t_2) \hat{\phi}_{2I} \ldots \hat{\phi}_{nI} \hat{U}(t_n, -\infty) | 0 \rangle \tag{228}$$

$$= \langle 0 | \hat{U}(\infty, t_1) \hat{\phi}_{1H} \hat{\phi}_{2H} \ldots \hat{\phi}_{nH} \hat{U}(t_0, -\infty) | 0 \rangle \tag{229}$$

Where $\hat{\phi}_H(t,x) = \hat{U}^\dagger(t,t_0) \hat{\phi}_I(t,x) \hat{U}(t,t_0)$
Now dealing with

$$\hat{U}(t_0, -\infty) | 0 \rangle = \lim_{t' \to -\infty} \lim_{t \to t_0} e^{i\hat{H}_0(t-t_0)} e^{-i\hat{H}(t-t')} e^{-i\hat{H}_0(t'-t)} | 0 \rangle \tag{230}$$

$$= \lim_{t' \to -\infty} \left(|\Omega\rangle \langle\Omega| + \sum_{n>0} e^{-iE_n(t_0-t')} |E_n\rangle \langle E_n| \right) | 0 \rangle \tag{231}$$

Where the nonvanishing terms are written in the eigenbasis of $\hat{H}$.

Quantum fields have a continuous spectra, such that $\sum_{n>0} \sim \int dE$.

Invoke the Riemann-Lebesgue Lemma ($\lim_{k\to\infty} \hat{\psi}(k) = \lim_{k\to\infty} \int \psi(x)e^{ikx}dx = 0$), and consider

$$\lim_{t'\to-\infty} \int dE \, \langle E|0 \rangle \, e^{-iE(t-t_0)} \, \langle \phi|E \rangle = 0 \tag{232}$$

The numerator is now equal to

$$\langle 0|\Omega \rangle \, \langle \Omega|0 \rangle \, \langle \Omega| \, \hat{\phi}_{1H} \ldots \hat{\phi}_{nH} \, |\Omega \rangle \tag{233}$$

Where $\langle 0|\Omega \rangle \, \langle \Omega|0 \rangle$ is equal to the denominator and cancels. QED.

Essentially, interacting quantum field theories come down to throwing in a Taylor series for the $\mathcal{S}$-matrix and $\hat{\phi}_{iI}$, and truncating some terms.

9 Lecture 9: Interactions and Feynman Diagrams

The story so far

- Built a free (non-interacting) relativistic quantum field theory, namely, the Klein-Gordon field, with Hamiltonian $\hat{H}_{KG}$.

- Added (Lorentz invaraint) interactions via the Hamiltonian

$$\hat{H}_{int} = \frac{\lambda}{4!} \int d^3x \, \hat{\phi}^4(\mathbf{x}), \, \lambda >> 1 \tag{234}$$

.

- Used perturbation theory to solve the Hamiltonian $\hat{H} = \hat{H}_{KG} + \hat{H}_{int}$.

- Studied observable quantities via the n-point correlation function

$$G^{(n)}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \langle \Omega | \, \mathcal{T}[\hat{\phi}_H(\mathbf{x}_1) \ldots \hat{\phi}_H(\mathbf{x}_n)] \, |\Omega\rangle \tag{235}$$

.

  Where $|\Omega\rangle$ is the full, interacting vacuum state, and $\hat{\phi}_H(\mathbf{x}_i) = \hat{\phi}_{iH} = \hat{\phi}_I(\mathbf{x}_i) = \hat{\phi}_{iI}$ is the field operator in the Heisenberg, interaction, picture, where the observables (e.g., field operators) closer to direct observation.

- Claimed and "proved" that the n-point correlation function can be calculated in terms of the ground state expectation values of the field operators, and the scattering $\mathcal{S}$-matrix .

$$G^{(n)}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \frac{\langle 0 | \, \mathcal{T}[\hat{\phi}_{1I} \ldots \hat{\phi}_{nI} \mathcal{S}] \, |0\rangle}{\langle 0 | \, \mathcal{S} \, |0\rangle} \tag{236}$$

  Where $\langle \phi | \, \mathcal{S} \, |\psi\rangle = \lim_{t \to \infty} \langle \phi | \, \mathcal{T}[e^{-i \int_{-t}^{t} \hat{H}_{int}(t')dt'}] \, |\psi\rangle, \, \forall \, |\phi\rangle, |\psi\rangle$

Now, to calculate quantities like the numerator of the n-point correlation function consider the field operator in the interaction picture (dropping the subscript "I")

$$\hat{\phi}_I(\mathbf{x}) = \hat{\phi}(\mathbf{x}) = \int \frac{d^3p}{\sqrt{2\omega_p}} (\hat{a}_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}} + \hat{a}_{\mathbf{p}}^\dagger e^{i\mathbf{p}\cdot\mathbf{x}}) = \hat{\phi}^+(\mathbf{x}) + \hat{\phi}^-(\mathbf{x}) \tag{237}$$

Where $\mathbf{p} = (\omega_p, p)$, with $\omega_p = \sqrt{p^2 + m^2}$, such that $\mathbf{p} \cdot \mathbf{x} = p^0 x^0 - p \cdot x$, and the newly defined operators annihilate the ground state, such that

$$\hat{\phi}^+(\mathbf{x}) |0\rangle = 0 \quad \text{and} \quad \langle 0 | \, \hat{\phi}^-(\mathbf{x}) = 0. \tag{238}$$

For example, consider the time-ordering of the two particle case (note the notation change for the four-vector in this section $\mathbf{x} \to x$)

$$
\begin{aligned}
\mathcal{T}[\hat{\phi}(x)\hat{\phi}(y)]_{x^0 > y^0} &= \hat{\phi}(x)\hat{\phi}(y) \\
&= \hat{\phi}^+(x)\hat{\phi}^+(y) + \hat{\phi}^+(x)\hat{\phi}^-(y) \\
&\quad + \hat{\phi}^-(x)\hat{\phi}^+(y) + \hat{\phi}^-(x)\hat{\phi}^-(y) \\
&= \hat{\phi}^+(x)\hat{\phi}^+(y) + \left(\hat{\phi}^-(y)\hat{\phi}^+(x) + [\hat{\phi}^+(x),\hat{\phi}^-(y)]\right) \\
&\quad + \hat{\phi}^-(x)\hat{\phi}^+(y) + \hat{\phi}^-(x)\hat{\phi}^-(y) \\
\mathcal{T}[\hat{\phi}(x)\hat{\phi}(y)]_{x^0 > y^0} &= \hat{\phi}^+(x)\hat{\phi}^+(y) + \left(\hat{\phi}^-(y)\hat{\phi}^+(x) + D(x-y)\cdot\mathbb{I}\right) \\
&\quad + \hat{\phi}^-(x)\hat{\phi}^+(y) + \hat{\phi}^-(x)\hat{\phi}^-(y)
\end{aligned}
$$

Then the ground state expectation value of two interacting field operators is simply the Feynman propagator

$$
\langle 0 |\, \mathcal{T}[\hat{\phi}(x)\hat{\phi}(y)]\, |0\rangle = \Delta_F(x-y) \tag{239}
$$

$$
= i \int \frac{d^4 p}{(2\pi)^4} \frac{e^{-ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon}\ ,\ \epsilon > 0 \tag{240}
$$

$$
= \begin{cases} D(x-y), & x^0 > y^0 \\ D(y-x), & x^0 \le y^0 \end{cases} \tag{241}
$$

Wick Contraction and Normal Ordering

Introduce some notation for extracting Feynman propagators from quantities like the expectation value of time-ordered field operators, called the **Wick contraction**. For two field operators, $\hat{\phi}(x)$ and $\hat{\phi}(y)$, and any three other operators $\hat{A}$, $\hat{B}$, and $\hat{C}$, write

$$
\overset{\frown}{\hat{\phi}(x)\hat{\phi}(y)} = \Delta_F(x-y)\cdot\mathbb{I} \tag{242}
$$

$$
A\overset{\frown}{\hat{\phi}(x)B\hat{\phi}(y)}C = \Delta_F(x-y)\cdot\hat{A}\hat{B}\hat{C} \tag{243}
$$

Also introduce **normal ordering**, denoted by $\mathcal{N}[\ ]$ that sends all "dagger" operators to the left. For example,

$$
\mathcal{N}[\hat{a}_p \hat{a}_q^\dagger \hat{a}_r \hat{a}_s^\dagger] = \hat{a}_q^\dagger \hat{a}_s^\dagger \hat{a}_p \hat{a}_r \tag{244}
$$

Observe the relationship between time-ordering and normal-ordering using the Wick contraction

$$
\mathcal{T}[\hat{\phi}(x)\hat{\phi}(y)] = \mathcal{N}[\hat{\phi}(x)\hat{\phi}(y) + \Delta_F(x-y)\cdot\mathbb{I}] \tag{245}
$$

$$
= \mathcal{N}[\hat{\phi}(x)\hat{\phi}(y) + \overset{\frown}{\hat{\phi}(x)\hat{\phi}(y)} \tag{246}
$$

A more involved example of the time-ordering of four field operators, where the only nonzero terms at the end of acting on states will be the "double contractions", since

$$\mathcal{T}[\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)\hat{\phi}(x_4)] = \mathcal{N}[\hat{\phi}_1\hat{\phi}_2\hat{\phi}_3\hat{\phi}_4 + \text{"all possible contractions"}]$$

$$= \mathcal{N}[\hat{\phi}_1\hat{\phi}_2\hat{\phi}_3\hat{\phi}_4 + \hat{\phi}_1\hat{\phi}_2\hat{\phi}_3\hat{\phi}_4 + \hat{\phi}_1\hat{\phi}_2\hat{\phi}_3\hat{\phi}_4$$

$$+ \hat{\phi}_1\hat{\phi}_2\hat{\phi}_3\hat{\phi}_4 + +\hat{\phi}_1\hat{\phi}_2\hat{\phi}_3\hat{\phi}_4 + \hat{\phi}_1\hat{\phi}_2\hat{\phi}_3\hat{\phi}_4 + \hat{\phi}_1\hat{\phi}_2\hat{\phi}_3\hat{\phi}_4$$

$$+ \hat{\phi}_1\hat{\phi}_2\hat{\phi}_3\hat{\phi}_4 + \hat{\phi}_1\hat{\phi}_2\hat{\phi}_3\hat{\phi}_4 + \hat{\phi}_1\hat{\phi}_2\hat{\phi}_3\hat{\phi}_4]$$

The ground state matrix elements of $\mathcal{T}[\hat{\phi}_1\hat{\phi}_2\hat{\phi}_3\hat{\phi}_4]$ is then

$$\langle 0| \mathcal{T}[\hat{\phi}_1\hat{\phi}_2\hat{\phi}_3\hat{\phi}_4] |0\rangle = \langle 0| \mathcal{N}[\hat{\phi}_1\hat{\phi}_2\hat{\phi}_3\hat{\phi}_4] |0\rangle + \langle 0| \mathcal{N}[\hat{\phi}_1\hat{\phi}_2\hat{\phi}_3\hat{\phi}_4] |0\rangle + \dots$$
$$+ \Delta_F(x_1 - x_2)\Delta_F(x_3 - x_4) + \Delta_F(x_1 - x_3)\Delta_F(x_2 - x_4)$$
$$+ \Delta_F(x_1 - x_4)\Delta_F(x_2 - x_3)$$

Where these are the values of the associated Feynman diagrams, which we write down in a "reverse" way, extracting the diagram from the calculated value. Later, we will extract the values from the diagrams.
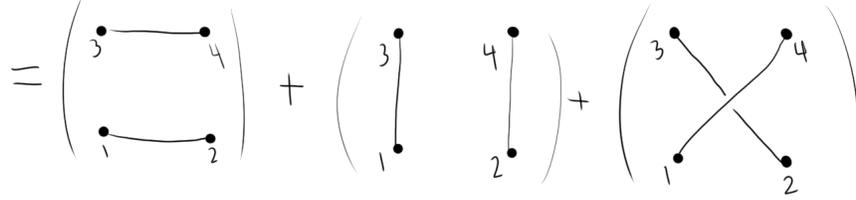
Figure 10: Feynman diagrams representing the nonzero values in the four particle example above.

Now stated in its general form

Wick's Theorem: $\mathcal{T}[\hat{\phi}_1 \dots \hat{\phi}_n] = \mathcal{N}[\hat{\phi}_1 \dots \hat{\phi}_n + \text{"all possible contractions"}]$.

Proof:
Induct on n, with the base case $n = 2$ confirmed to be true, and check that the $n - 1$ case implies the full n case.
Assume, without loss of generality, that everything is time-ordered, such that $x_1^0 > x_2^0 > \dots > x_n^0$.
Then the left-hand side of Wick's theorem becomes

$$\mathcal{T}[\hat{\phi}_1 \dots \hat{\phi}_n] = \hat{\phi}_1\hat{\phi}_2 \dots \hat{\phi}_n. \tag{247}$$

Use the inductive hypothesis for $n-1$ case on this equation

$$\hat{\phi}_1\hat{\phi}_2\ldots\hat{\phi}_n = (\hat{\phi}_1^+ + \hat{\phi}_1^-)\mathcal{N}[\hat{\phi}_2\ldots\hat{\phi}_n + \begin{pmatrix}\text{all possible contractions}\\ \text{excluding } \hat{\phi}_1.\end{pmatrix}]$$

$$= \hat{\phi}_1^+\mathcal{N}[\hat{\phi}_2\ldots\hat{\phi}_n + \begin{pmatrix}\text{all possible contractions}\\ \text{excluding } \hat{\phi}_1.\end{pmatrix}]$$

$$+ \mathcal{N}[\hat{\phi}_1^-\hat{\phi}_2\ldots\hat{\phi}_n + \hat{\phi}_1^-\begin{pmatrix}\text{all possible contractions}\\ \text{excluding } \hat{\phi}_1.\end{pmatrix}].$$

Since $\hat{\phi}_1^-$ is already normal-ordered.
Focus on the first part of the first term of $\hat{\phi}_1\hat{\phi}_2\ldots\hat{\phi}_n$ above

$$\hat{\phi}_1^+\mathcal{N}[\hat{\phi}_2\ldots\hat{\phi}_n] = \mathcal{N}[\hat{\phi}_2\ldots\hat{\phi}_n]\hat{\phi}_1^+ + [\hat{\phi}_1^+,\mathcal{N}[\hat{\phi}_2\ldots\hat{\phi}_n]]$$

$$= \mathcal{N}[\hat{\phi}_1^+\hat{\phi}_2\ldots\hat{\phi}_n]$$

$$+ \mathcal{N}[[\hat{\phi}_1^+,\hat{\phi}_2]\hat{\phi}_3\ldots\hat{\phi}_n + \hat{\phi}_2[\hat{\phi}_1^+,\hat{\phi}_3]\hat{\phi}_4\ldots\hat{\phi}_n + \cdots + \hat{\phi}_2\hat{\phi}_3\ldots\hat{\phi}_{n-1}[\hat{\phi}_1^+,\hat{\phi}_n]]$$

$$\hat{\phi}_1^+\mathcal{N}[\hat{\phi}_2\ldots\hat{\phi}_n] = \mathcal{N}[\hat{\phi}_1^+\hat{\phi}_2\ldots\hat{\phi}_n + \overset{\lceil\quad\rceil}{\hat{\phi}_1^+\hat{\phi}_2}\hat{\phi}_3\ldots\hat{\phi}_n + \hat{\phi}_1^+\hat{\phi}_2\overset{\lceil\quad\rceil}{\hat{\phi}_3}\ldots\hat{\phi}_n + \cdots + \hat{\phi}_1^+\hat{\phi}_2\hat{\phi}_3\ldots\overset{\lceil\qquad\quad\rceil}{\hat{\phi}_n}].$$

Where the second equality follows from $[\hat{\phi}_j^\dagger,\hat{\phi}_k] \propto \mathbb{I}$.
Now, focus on the rest of the first term of $\hat{\phi}_1\mathcal{N}[\hat{\phi}_2\ldots\hat{\phi}_n]$

$$\hat{\phi}_1^+\mathcal{N}[\begin{pmatrix}\text{all possible contractions}\\ \text{excluding } \hat{\phi}_1.\end{pmatrix}] = [\hat{\phi}_1^+,\mathcal{N}[\ldots]] + \mathcal{N}[\ldots]\hat{\phi}_1^+$$

$$= \mathcal{N}[\begin{pmatrix}\text{all possible contractions}\\ \text{including } \hat{\phi}_1^+.\end{pmatrix}] + \mathcal{N}[\hat{\phi}_1^+\begin{pmatrix}\text{all possible contractions}\\ \text{excluding } \hat{\phi}_1.\end{pmatrix}]$$

10 Lecture 10: Feynman Rules for φ^4 Theory

In perturbative, interacting field theories, we must calculate the n-point correlation function

$$G^{(n)}(x_1, \ldots, x_n) = \frac{\langle 0| \, \mathcal{T}[\hat{\phi}_1 \ldots \hat{\phi}_n \mathcal{S}] \, |0\rangle}{\langle 0| \, \mathcal{S} \, |0\rangle} \tag{248}$$

Where the scattering matrix elements and the interacting Hamiltonian are

$$\langle \phi | \, \mathcal{S} \, | \psi \rangle = \lim_{t \to \infty} \langle \phi | \, \mathcal{T}[e^{-i \int_{-t}^{t} \hat{H}_I(t') dt'}] \, | \psi \rangle \tag{249}$$

$$\hat{H}_I(t) = \frac{\lambda}{4!} \int d^3 x \, \hat{\phi}^4(\mathbf{x}) \tag{250}$$

The numerator and denominator of the correlation function must be expanded perturbatively in the small parameter λ.

$$G^{(n)} = \frac{a_0 + \lambda a_1 + \lambda^2 a_2 + \ldots}{b_0 + \lambda b_1 + \lambda^2 b_2 + \ldots}, \text{ where } a_j, b_j \in \mathbb{C} \tag{251}$$

$$= \frac{1}{b_0}(a_0 + \lambda a_1 + \ldots)(1 - \lambda \frac{b_1}{b_0} + \lambda^2 \frac{b_2}{b_0} + \ldots) \tag{252}$$

$$=_{\mathcal{O}(\lambda)} \frac{1}{b_0}(a_0 + \lambda a_1)(1 - \lambda \frac{b_1}{b_0}) \tag{253}$$

$$G^{(n)} = \frac{a_0}{b_0} + \lambda(\frac{a_1}{b_0} - \frac{b_1 a_0}{b_0^2}) \text{ to order } \lambda \tag{254}$$

This gives the intermediate task of calculating the coefficients of the perturbative expansion: a_0, a_1, b_0, and b_1.

Consider the $n = 2$ case

$$G^{(2)}(x, y) = \lim_{t \to \infty} \frac{\langle 0| \, \mathcal{T}[\hat{\phi}(x)\hat{\phi}(y) e^{-i \int_{-t}^{t} \hat{H}_I(t') dt'}] \, |0\rangle}{\langle 0| \, \mathcal{T}[e^{-i \int_{-t}^{t} \hat{H}_I(t') dt'}] \, |0\rangle} \tag{255}$$

Expand the exponentials, keeping to order λ, and insert into numerator. The numerator is then

$$\langle 0| \, \mathcal{T}[\hat{\phi}(x)\hat{\phi}(y)\mathcal{S}] \, |0\rangle = \langle 0| \, \mathcal{T}[\hat{\phi}(x)\hat{\phi}(y) e^{-i \int_{-t}^{t} \hat{H}_I(t') dt'}] \, |0\rangle \tag{256}$$

$$= \langle 0| \, \mathcal{T}[\hat{\phi}(x)\hat{\phi}(y) \left(\mathbb{I} - \frac{i\lambda}{4!} \int d^4 z \, \hat{\phi}^4(z) \right)] \, |0\rangle \tag{257}$$

$$= \langle 0| \, \mathcal{T}[\hat{\phi}(x)\hat{\phi}(y)] \, |0\rangle - \frac{i\lambda}{4!} \int d^4 z \, \langle 0| \, \mathcal{T}[\hat{\phi}(x)\hat{\phi}(y)\hat{\phi}^4(z)] \, |0\rangle \tag{258}$$

$$\langle 0| \, \mathcal{T}[\hat{\phi}(x)\hat{\phi}(y)\mathcal{S}] \, |0\rangle = \Delta_F(x - y) - \frac{i\lambda}{4!} \int d^4 z \, \langle 0| \mathcal{N}[\hat{\phi}(x)\hat{\phi}(y)\hat{\phi}^4(z) + \text{all contractions}] \, |0\rangle \tag{259}$$

The last line is gotten by applying Wick's theorem, and recall that any terms with *uncontracted* operators will evaluate to **zero** in the vacuum expectation value (e.g., $\langle 0 | \mathcal{N}[\ldots] | 0 \rangle$), and only the *fully contracted* terms will contribute, such as $\langle 0 | \hat{\phi}(x)\hat{\phi}(y)\hat{\phi}(z)\hat{\phi}(z)\hat{\phi}(z)\hat{\phi}(z) | 0 \rangle$, and $\langle 0 | \hat{\phi}(x)\hat{\phi}(y)\hat{\phi}(z)\hat{\phi}(z)\hat{\phi}(z)\hat{\phi}(z) | 0 \rangle$.

Now, "all good physics and math ends in linear algebra, combinatorics, or both", and we use combinatorics to calculate how many fully contracted terms we expect to see in the expansion.

- There are $(2n - 1)!!$ full contractions per expansion, where n is the number of unique particles. In this case, $n = 3$ for x, y, and z spacetime coordinates.

- There are 2 unique contraction types out of the $(2 \cdot 3 - 1)!! = 15$ full contractions.

$$\Delta_F(x - y)\Delta_F^2(z - z)$$
$$\Delta_F(x - z)\Delta_F(y - z)\Delta_F(z - z)$$

- There are 3 contractions of the first type

  Connect $x \to y$ in 1 way and $z \to z$ twice, each in only 1 way.

- There are 12 contractions of the second type

  Connect $x \to z$ in 4 ways, followed by $y \to z$ in 3 ways, and $4 \cdot 3 = 12$.

Return to the numerator of the correlation function, and pretending that $\Delta_F(z - z)$ is finite, for now,

$$\langle 0 | \mathcal{T}[\hat{\phi}(x)\hat{\phi}(y)\mathcal{S}] | 0 \rangle = \Delta_F(x - y)$$

$$+ \left(-\frac{i\lambda}{4!} \right) \int d^4z \ \langle 0 | \mathcal{N}[\hat{\phi}(x)\hat{\phi}(y)\hat{\phi}^4(z) + \underset{\substack{\text{all partial} \\ \text{contractions}}}{} + \underset{\substack{\text{all full} \\ \text{contractions}}}{}] | 0 \rangle$$

$$= \Delta_F(x - y) + 3 \cdot \left(-\frac{i\lambda}{4!} \right) \int d^4z \, \Delta_F(x - y)\Delta_F^2(z - z)$$

$$+ 12 \cdot \left(-\frac{i\lambda}{4!} \right) \int d^4z \, \Delta_F(x - z)\Delta_F(y - z)\Delta_F(z - z)$$

And in diagrammatic form

Figure 11: Feynman diagram representation of the above integral values for the φ^4 interacting theory.

Similarly calculate the denominator of the two-point correlation function to obtain the following

$$G^{(2)}(x,y) = \frac{\overset{x \quad y}{\bullet\!-\!\!-\!\bullet} \; + 3\left(\underset{x \quad y}{\bullet\!-\!\!-\!\bullet} \; \underset{z}{\bigotimes} \right) + 12 \; \underset{x \quad z \quad y}{\overset{\bigcirc}{\bullet\!-\!\!\bullet\!-\!\!\bullet}}}{1 \; + \; 3 \; \underset{z}{\bigotimes}}$$

$$= \left(\bullet\!-\!\bullet \; + 3\left(\bullet\!-\!\bullet \; \bigotimes \right) + 12 \; \overset{\bigcirc}{\bullet\!-\!\!\bullet\!-\!\!\bullet} \right) \cdot \left(1 - 3 \, \bigotimes \right)$$

$$\overset{\vee}{=} \; \bullet\!-\!\bullet \; + \; 3\left(\cancel{\bullet\!-\!\bullet\!\bigotimes} \right) - 3 \left(\cancel{\bullet\!-\!\bullet\!\bigotimes} \right) + 12 \; \overset{\bigcirc}{\bullet\!-\!\!\bullet\!-\!\!\bullet}$$

$$= \; \bullet\!-\!\!-\!\bullet \; + \; 12 \; \overset{\bigcirc}{\bullet\!-\!\!\bullet\!-\!\!\bullet}$$

$$G^{(2)}(x,y) = \Delta_F(x-y) - \frac{12 i \lambda}{4!} \int d^4z \; \Delta_F(x-z) \Delta_F(y-z) \Delta_F(z-z)$$

Figure 12: Feynman diagram calculation for the two-point correlation function for the φ^4 theory. Note that this "miraculous" cancellation of divergent terms (the "self-interacting figure-eight") actually follows from a more general result.

The arguments of the correlation function are the **external vertices**, and the spacetime coordinates of the interacting Hamiltonian are the **internal vertices**. To calculate the value of a diagram, associate a factor, the Feynman propagator, $\Delta_F(x - y)$ to each edge connecting external vertices (e.g., x and y), and associate the factor $-i\lambda \int d^4z$ (dividing by the symmetry factor 4!) to each internal vertex (e.g., z).

For example,

$$\equiv \left(-\frac{i\lambda}{4!}\right)^4 \int\int\int\int d^4 z_1 d^4 z_2 d^4 z_3 d^4 z_4 \times$$

$$\times \Big(\Delta_f(y-z_4)\Delta_f(x-z_4)\Delta_f(z_1-z_4)\Delta_f(z_1-z_4)\times$$

$$\times \Delta_f(z_1-z_2)\Delta_f(z-z_1)\Delta_f(w-z_2)\times$$

$$\times \Delta_f(z_2-z_3)\Delta_f(z_1-z_3)\Delta_f(z_3-z_3)\Big)$$
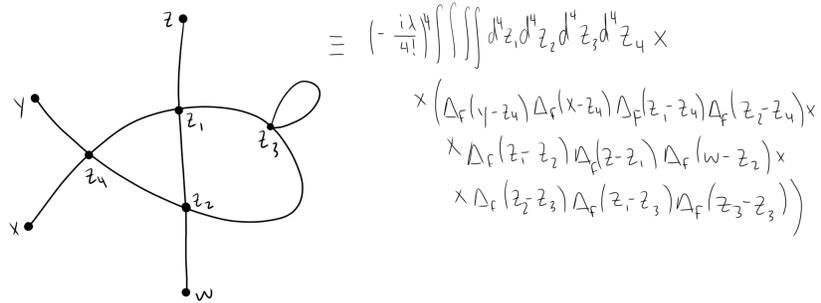
Figure 13: Example Feynman diagram and associated values with 4 external vertices and 4 internal vertices.

In summary, to calculate $G^{(2)}(x,y)$ to arbitrary order, sum over all possible diagrams with 2 external vertices, subject to the (canonical) Feynman rules for φ^4 theory.

$$G^{(2)}(x,y) = \frac{\langle 0| \mathcal{T}[\hat{\phi}(x)\hat{\phi}(y)\mathcal{S}] |0\rangle}{\langle 0| \mathcal{S} |0\rangle} = \begin{pmatrix} \text{sum of all possible diagrams} \\ \text{with two external vertices} \end{pmatrix} \qquad (260)$$

Figure 14: **Canonical Feynman rules for φ^4 theory**.

Note the abscence of the inverse 4! factor in rule number 2, as it is added based on observation, and not considered a component on the *canonical* rules.

Higher order terms in the expansion of the two-point correlation function become successively more complicated and often redundant. For example, The self-interacting "figure-eight" diagram of an internal vertex can occur in $4! \cdot 8 = 192$ ways. The redundancy is encoded in the **symmetry factor** (see rule 4 above) of the diagram. Typically, in practice and computation, distinct diagrams are written down and overcounting is determined via the symmetry factor.

11 Lecture 11: Feynman Rules and Vacuum Bubbles

The Feynman rules demonstrated in the last lecture are specifically for the φ^4 theory of interacting quantum fields in *position space*, and allow the calculation of n-point correlation (Green's) functions, which are not directly observable, but related to scattering amplitudes which are directly observable. We did not rigorously prove, but demonstrated (with $n = 2$) the feasibility of and accepted as "definition", that the n-point correlation function is equal to the sum of all possible diagrams with n external vertices, subject to the Feynman rules in position space. Keep note that we did not consider $n > 2$ or $\lambda > 1$ in the following equality.

$$G^{(n)}(x_1, \ldots, x_n) = \frac{\langle 0 | \mathcal{T}[\hat{\phi}(x_1) \ldots \hat{\phi}(x_n) \mathcal{S}] | 0 \rangle}{\langle 0 | \mathcal{S} | 0 \rangle} \tag{261}$$

$$= \begin{pmatrix} \text{sum of all possible diagrams with } n \text{ external vertices} \\ \text{subject to Feynman rules in position space} \end{pmatrix} \tag{262}$$

A ubiquitous issue and deep concern in quantum field theory is the appearance infinities in calculations. Nature seems to suggest that there are no infinities, unless one asks the wrong question. (Are there actually any physical quantities that can be proven to be infinite by experiment, such as the energy levels of the hydrogen atom or the results of continuous scattering theory?) Many infinities that appear due to the application of the Feynman rules will be trivially dispelled (*cf.*, in quantum mechanics when an infinite ground state energy is calculated, simply apply a shift to make it finite). **Vacuum bubbles** are diagrams with no external vertices (e.g., self interactions) that evaluate to infinity, but will cancel in calculating the n-point correlation function $G^{(n)}(x_1, \ldots, x_n)$, as we saw in the $n = 2$ case in the last lecture.

Feynman Rules in Momentum Space (φ^4 Theory)

Remember that the perturbation expansion of the $\mathcal{S}$-matrix and the Green's function can be calculated in momentum space via the Feynman rules in momentum space. The momentum space Feynman propagator has the form

$$\Delta_F(x, y) = \int \frac{d^4 p}{(2\pi)^4} \frac{i e^{-ip \cdot (x-y)}}{p^2 - m^2 + i\epsilon} \tag{263}$$

Calculating the perturbation expansion in momentum space may make additional cancellations more apparent. The Feynman rules for φ^4 theory in momentum space are as follows

1. For each propagator with momentum p, add a factor of $\frac{i}{p^2 - m^2 + i\epsilon}$.

2. For each (internal) vertex, add a factor of $-i\lambda$.

3. For each external vertex, add a factor of $e^{-ip \cdot x}$.

4. Impose momentum conservation at each vertex.

5. Integrate over undetermined momenta.

6. Divide by symmetry factor.

$$\text{propagator} \quad \xrightarrow{\vec{p}} \quad = \quad \frac{i}{p^2 - m^2 + i\epsilon}$$

$$\begin{array}{c}(\text{internal}) \\ \text{vertex}\end{array} \quad \overset{\vec{p_j}}{\underset{(j=1,2,3,4)}{\times\!\!\times}} \quad = \quad -i\lambda$$

$$\begin{array}{c}(\text{external}) \\ \text{point}\end{array} \quad \overset{\vec{p}}{\times\!\!\longleftarrow} \quad = \quad e^{-ipx}$$

$$\underline{\text{note:}} \quad \xrightarrow{\vec{p}} \quad = \quad \overset{-\vec{p}}{\longleftarrow}$$
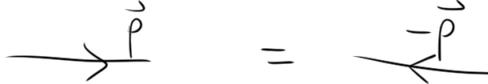
Figure 15: Diagrammatic representation for the Feynman rules in momentum space.

Returning to the $n = 2$ case, a general diagram consists of a product of *connected* components and *disconnected* components. Note that the external vertices (x and y in the $n = 2$ case) are always connected, since their degrees are odd. For example,
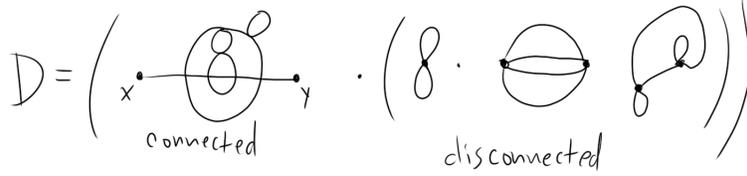
Figure 16: Typical diagram for $n = 2$ case.

All possible disconnected pieces form a countable set.
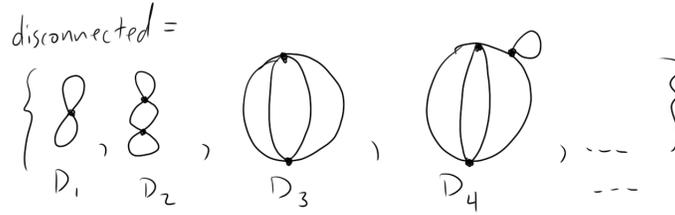
Figure 17: Countable set of all possible disconnected pieces.

Each type of disconnected diagram D_j has a unique value $v(D_j) = v_j$, which are all *infinite*. To interpret this result, place a **cutoff** on the theory. It must be checked that the results, after evaluating the diagrams of the expansion, do not depend on the cutoff imposed.

Suppose that a given diagram D has n_j components of type D_j, including the connected component. The value of the diagram is the product of the values of the connected and disconnected components.

$$v_j = v_{connected} \cdot \prod_{j=1}^{\infty} \frac{1}{n_j!} v_j^{n_j} \tag{264}$$

Where $n_j!$ is the symmetry factor.

For the $n = 2$ diagram example above, we have the product of the connected piece with disconnected types D_1, D_3, and D_2, respectively, with $n_j = 1$ for each diagram type j. Therefore the value of the diagram is

$$v(D) = v_{connected} \cdot v_1 \cdot v_3 \cdot v_2 \tag{265}$$

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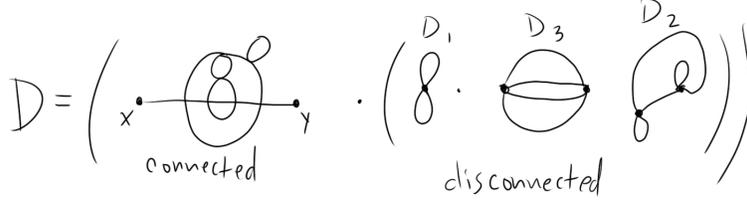
Figure 18: Typical diagram for $n = 2$ case with disconnected pieces labelled by type.

This allows us to write down a more closed form for the series expansion for $G^{(2)}(x, y)$, and ditch the "all possible diagrams" bit. The numerator of $G^{(2)}(x, y)$ is then

$$\text{Numerator}(G^{(2)}(x, y)) = \sum_{connected} \sum_{\{n_j\}} v_{connected} \cdot \prod_{j=1}^{\infty} \frac{1}{n_j!} v_j^{n_j} \tag{266}$$

$$= \sum_{conn.} v_{conn.} \sum_{\{n_j\}} \prod_{j=1}^{\infty} \frac{1}{n_j!} v_j^{n_j} \tag{267}$$

$$= \sum_{conn.} v_{conn.} \prod_{j=1}^{\infty} \sum_{\{n_j\}} \frac{1}{n_j!} v_j^{n_j} \tag{268}$$

$$\text{Numerator}(G^{(2)}(x, y)) = \sum_{conn.} v_{conn.} \cdot e^{\sum_{j=1}^{\infty} v_j} \tag{269}$$

Note that we justify these manipulations of not-necessarily-convergent series by the implemented cutoff, which makes each value finite. Similarly, the denominator is simply the sum over the same exponential

$$\text{Denominator}(G^{(2)}(x, y)) = e^{\sum_{j=1}^{\infty} v_j} \tag{270}$$

This cancels exactly with the same term in the numerator, pulled out of the sum over connected parts. Therefore, the 2-point correlation function, which generalizes to $n > 2$, is equal to the sum of all connected components subject to the Feynman rules.

$$G^{(2)}(x, y) = \begin{pmatrix} \text{sum over all connected diagrams} \\ \text{subject to Feynman rules} \end{pmatrix} \tag{271}$$

$$G^{(2)}(x,y) = \begin{pmatrix} \text{sum over all connected diagrams} \\ \text{subject to the Feynman rules} \end{pmatrix}$$

$$\left(\underline{note} \text{: same for } n > 2 \right)$$

Figure 19: The sum over all connected diagrams subject to the Feynman rules.

Cutoffs in QFT

Consider the Klein-Gordon Hamiltonian

$$\hat{H}_{KG} = \frac{1}{2} \int d^3x \left(\hat{\pi}^2(x) + (\nabla \hat{\phi}(x))^2 + m^2 \hat{\phi}^2(x) \right) \tag{272}$$

This assumes an infinitie number of degrees of freedom, one per each point in spacetime $x \in \mathcal{M}_4$. Now, for example, our theory and understanding of spacetime breaks down below the Planck scale. We can either continue to work in ignorance, or remember that we are dealing in *effective* theories that, hopefully, represent something more fundamental. For example, the Navier-Stokes equation is an effective theory for quantum chromodynamics (QCD). An **effective theory** is a description which explains all observations up to a given scale s, often expressed in inverse length, and may break down beyond the given scale.

To impose a cutoff on the Hamiltonian, add a "fixing" term $\hat{H}_\Lambda$ which is well-behaved up to the scale Λ, and must match predictions of the original Hamiltonian $\hat{H}$

$$\hat{H} \to \hat{H}' = \hat{H} + \hat{H}_\Lambda \tag{273}$$

To modify the new Hamitonian for other scales, Wilson's **renormalization theory** may be used, where the "fixing" Hamiltonian $\hat{H}_\Lambda$ is parameterized by

the field operators and their derivatives

$$\hat{H}_\Lambda = \sum_{polynomials} \mathcal{P}(\hat{\phi}, \nabla\hat{\phi}, \nabla^2\hat{\phi}, \ldots, \hat{\pi}, \nabla\hat{\pi}, \nabla^2\hat{\pi}, \ldots) \tag{274}$$

$$= c_0\hat{\phi} + c_1\hat{\phi}^2 + \cdots + d_0\hat{\pi} + d_1\hat{\pi}^2 + \cdots + e_1(\nabla\hat{\phi})^2 + \ldots \tag{275}$$

Only a finite number of the coefficients are nonzero and have an effect on observations beyond the scale Λ. By effect, the value of the term in the sum over field operator polynomials is not suppressed by inverse powers of Λ. Such terms that are not suppressed and have an effect on observations are called **relevant terms**. Any cutoff that is imposed must differ only by relevant terms.

For example, the relevant terms up to 4 dimensions is

$$\{\hat{\phi}, \hat{\phi}^2, \nabla^2\hat{\phi}, \hat{\pi}, \hat{\pi}^2, \hat{\phi}^3, \hat{\phi}^4\}. \tag{276}$$

The simplest cutoff to impose on the momentum states of the free Klein-Gordon Hamiltonian is

$$\hat{H}_{KG} = \int \frac{d^3p}{(2\pi)^3}\omega_p\hat{a}_p^\dagger\hat{a}_p \rightarrow \hat{H}_{KG} + \hat{H}_\Lambda = \int_{|p|<\Lambda} \frac{d^3p}{(2\pi)^3}\omega_p\hat{a}_p^\dagger\hat{a}_p \tag{277}$$

And the simplest cutoff for φ^4 interaction is

$$\hat{H}_{\varphi^4} = \frac{\lambda}{4!}\int d^3x\,\hat{\phi}^4(x) \rightarrow \hat{H}_{\varphi^4} + \hat{H}_\Lambda = \int_{|p_j|<\Lambda} d^3p_1 d^3p_2 d^3p_3 d^3p_4(\ldots) \tag{278}$$

The new Hamiltonian for the quantum Klein-Gordon field with φ^4 interactions, and an imposed cutoff at scale Λ is then written as

$$\hat{H} \rightarrow \hat{H}' = \hat{H}_{KG} + \hat{H}_{\varphi^4} + \hat{H}_\Lambda \tag{279}$$

$$= \int_{|p|<\Lambda} \frac{d^3p}{(2\pi)^3}\omega_p\hat{a}_p^\dagger\hat{a}_p + \int_{|p_j|<\Lambda} d^3p_1 d^3p_2 d^3p_3 d^3p_4(\ldots) \tag{280}$$

12 Lecture 12: The $\mathcal{S}$-matrix in φ^4 Theory

Recall that some diagrams evaluate to infinity in the calculation of the time-ordered n-point correlation function

$$G^{(n)}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \langle\Omega|\, \mathcal{T}[\hat{\phi}(\mathbf{x}_1)\ldots\hat{\phi}(\mathbf{x}_n)]\,|\Omega\rangle = \begin{pmatrix}\text{\scriptsize sum over all connected diagrams}\\ \text{\scriptsize with } n \text{ external points/legs}\end{pmatrix} \tag{281}$$

Example: $n = 4$ Diagrams

For example, the $n = 4$ case sums over diagrams like the following
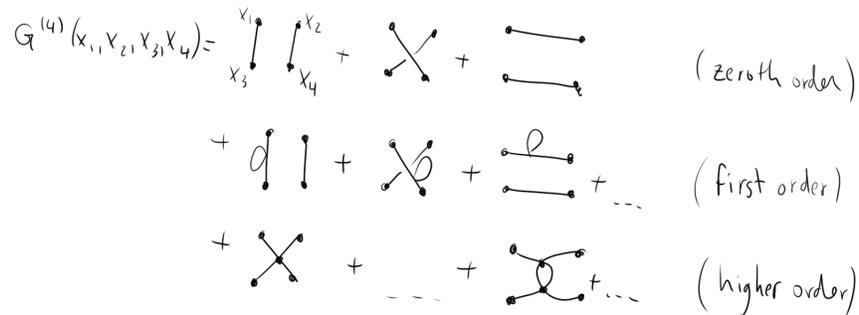
Figure 20: Sum over connected diagrams for $n = 4$ case.

Diagrams with one loop (*first order*) evaluate to infinity, but are easily eschewed. Two vertex diagrams (last one in figure above) is a much more difficult infinity to tame, and causes divergences, typically of the form, in momentum space,

$$\mathcal{I} = \int \frac{d^4p}{(2\pi)^4} \frac{i}{\mathbf{p}^2 - m^2 + i\epsilon} \tag{282}$$

To attempt to tame the infinity, impose a cutoff at scale Λ

$$\mathcal{I} \to \mathcal{I}(\Lambda) = \int_{|p|<\Lambda} \frac{d^4p}{(2\pi)^4} \frac{i}{\mathbf{p}^2 - m^2 + i\epsilon}. \tag{283}$$

To cope with arbitrary choices of Λ, arbitrary coupling constants can be employed to compensate and remove possible Λ-dependencies in calculations. Further explanation can be found in the renormalization theory of scalar particles.

Scattering Theory

Scattering theory is an effective theory for large time limits $|t| \gg$ constant. The probability of a scattering event occuring, the scattering amplitude, in the

case of well-collimated beams with incoming momenta $\mathbf{k}_A$ and $\mathbf{k}_B$, and outgoing momenta $\mathbf{p}_j$, $j = 1, \ldots, n$, is related to the quantity

$$_{out} \langle \mathbf{p}_1 \mathbf{p}_2 \ldots \mathbf{p}_n | \mathbf{k}_A \mathbf{k}_B \rangle_{in} = \langle \mathbf{p}_1 \mathbf{p}_2 \ldots \mathbf{p}_n | \mathcal{S} | \mathbf{k}_A \mathbf{k}_B \rangle . \tag{284}$$
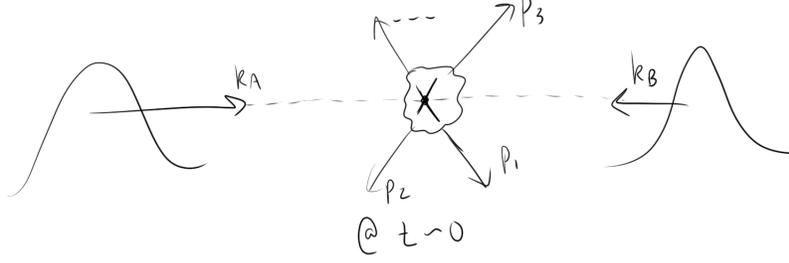

Figure 21: Schematic of scattering experiment with incoming momenta and outgoing momenta as described above.

For the free, non-interacting, theory, we have the equation for incoming momenta related to the free vacuum state, denoted by $| \rangle_0$,

$$|k_A k_B\rangle_{in} = \sqrt{2\omega_{k_A} \omega_{k_B}} \, \hat{a}^\dagger_{k_A} \hat{a}^\dagger_{k_B} |0\rangle_0 . \tag{285}$$

The task at hand is to calculate the scattering amplitude $\langle \mathbf{p}_1 \mathbf{p}_2 \ldots \mathbf{p}_n | \mathcal{S} | \mathbf{k}_A \mathbf{k}_B \rangle$. Define the $\mathcal{S}$-matrix as

$$\mathcal{S} = \mathbb{I} + i\hat{\mathbb{T}} . \tag{286}$$

Where the identity $\mathbb{I}$ is the dominant term in small intercation, where almost nothing happens, and the operator $\hat{\mathcal{T}}$ is the dominant in larger interactions. Introduce the quantity $\mathcal{M}$ to factor out the conservation of momentum in the scattering process, where all momenta are "on shell", such that $\mathbf{p}^0 = \omega_p$.

$$\langle \mathbf{p}_1 \ldots \mathbf{p}_n | i\hat{\mathbb{T}} | \mathbf{k}_A \mathbf{k}_B \rangle = (2\pi)^4 \delta^{(4)}\left((\mathbf{k}_A + \mathbf{k}_B) - \sum_{j=1}^{n} p_j \right) \cdot i\mathcal{M}((\mathbf{k}_A, \mathbf{k}_B) \to \mathbf{p}_f) . \tag{287}$$

Recall that the differential scattering cross section of $2 \to 2$ particles in the center of mass frame is

$$\left(\frac{d\sigma}{d\Omega} \right)_{CM} = \frac{|\mathcal{M}|^2}{64\pi^2 E_{CM}^2} \tag{288}$$

To compute the scattering amplitude $\langle \mathbf{p}_1 \mathbf{p}_2 \ldots \mathbf{p}_n | \mathcal{S} | \mathbf{k}_A \mathbf{k}_B \rangle$, take "on faith" that for small interactions

$$|k_A k_B\rangle_{int} \propto \lim_{t \to \infty} e^{-i\hat{H}t} |k_A k_B\rangle_0 . \tag{289}$$

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And compare this to the result related to the Riemann-Lebesgue lemma

$$|\Omega\rangle_{int} = \lim_{t \to \infty} e^{-i\hat{H}t} |0\rangle_0 \,. \tag{290}$$

An argument for the plausibility for the second equality of $|\Omega\rangle_{int}$, where $\hat{H} = \hat{H}_{KG} + \hat{H}_{int}$, $\hat{H}_{KG} |0\rangle_0 = 0$, and $\hat{H} |\Omega\rangle_{int} = 0$, is as follows. Suppose that the Hamiltonian is parameterized by $s \in [0,1]$, such that $\hat{H}(s) = \hat{H}_{KG} + s\hat{H}_{int}$, and that the free Hamiltonian $\hat{H}_{KG}$ has a spectral gap of $\Delta(s) = m$. When $s = 0$, we have the free, Klein-Gordon Hamiltonian, and when $s = 1$, we have the interacting, φ^4 Hamiltonian. Then it's plausible that the spectral gap in the spectrum of eigenvalues of $\hat{H}(s)$ will always be nonzero for at least just one eigenvalue, and the spectrum is adiabatically connected (can't instantaneously go from massful to massless, but interactions can be turned on/off gradually).


Figure 22: Schematic of the spectral gap for the parameterized Hamiltonian.

For the proportionality factor of $|k_A k_B\rangle_{int}$ to be true for small interactions, which is far more radical, it is required that there are nonzero spectral gaps, for some $s > 0$, at each momentum eigenstate of momenta $\mathbf{k}_A$ and $\mathbf{k}_B$, and there are no other nearby eigenvalues that may be mistaken for the incoming momenta. In practice, these incoming particles can create bound states, for example, and make this distinction difficult. If all of this is justifiable, and we accept the proportionality $|k_A k_B\rangle_{int} \propto \lim_{t \to \infty} e^{-i\hat{H}t} |k_A k_B\rangle_0$, then we can create equalities and proportionalities, although difficult, to quantities that we can actually calculate

$$\lim_{t \to \infty} {}_0\langle \mathbf{p}_1 \cdots \mathbf{p}_n| e^{-ihatH \cdot 2t} |\mathbf{k}_A \mathbf{k}_B\rangle_0 \propto \lim_{t \to \infty} {}_0\langle \mathbf{p}_1 \cdots \mathbf{p}_n| \mathcal{T}[e^{-i\int_{-t}^{t} \hat{H}_{int}(t')dt'}] |\mathbf{k}_A \mathbf{k}_B\rangle_0 \tag{291}$$

This is exactly the same as the argument for the Dyson series expansion for $G^{(n)}$ with the vacuum states, and, just as in that case, the proportionality difficulty can be eliminated, and we have the following (not yet justified) equality

$$\langle \mathbf{p}_1 \cdots \mathbf{p}_n| i\hat{\mathbb{T}} |\mathbf{k}_A \mathbf{k}_B\rangle = \lim_{t \to \infty} \left({}_0\langle \mathbf{p}_1 \cdots \mathbf{p}_n| \mathcal{T}[e^{-i\int_{-t}^{t} \hat{H}_{int}(t')dt'}] |\mathbf{k}_A \mathbf{k}_B\rangle_0 \right)_{\substack{\text{connected,} \\ \text{amputated}}} \tag{292}$$

The condition of "connected, amputated" is analogous to "connected" as in the $G^{(n)}$ case.

We can begin to justify this equality through some calculations. Consider the $\mathcal{O}(1)$ term, corresponding to the identity operator in $\mathcal{S} = \mathbb{I} + i\hat{\mathbb{T}}$, for the $2 \to 2$ scattering process.

$$_{out}\langle \mathbf{p}_1\mathbf{p}_2 | \mathbf{k}_A\mathbf{k}_B \rangle_{in} =_{\mathcal{O}(1)} {}_0\langle \mathbf{p}_1\mathbf{p}_2 | \mathbf{k}_A\mathbf{k}_B \rangle_0 \tag{293}$$

$$= \sqrt{2\omega_{p_1}\omega_{p_2} \cdot 2\omega_{k_A}\omega_{k_B}} \langle 0 | \hat{a}_{p_1}\hat{a}_{p_2}\hat{a}_{k_A}^\dagger \hat{a}_{k_B}^\dagger | 0 \rangle \tag{294}$$

$$= 2\omega_A \cdot 2\omega_B (2\pi)^6 (\delta(\mathbf{p}_1 - \mathbf{k}_A)\delta(\mathbf{p}_2 - \mathbf{k}_B) + \delta(\mathbf{p}_1 - \mathbf{k}_B)\delta(\mathbf{p}_2 - \mathbf{k}_A)) \tag{295}$$
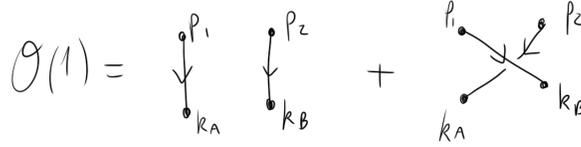

Figure 23: Diagram of the $\mathcal{O}(1)$ term as calculated above.

The next term to $\mathcal{O}(\lambda)$, applying Wick's theorem, and retaining all contractions, since we are not just working with the vacuum state anymore,

$$\langle \mathbf{p}_1\mathbf{p}_2 | i\hat{\mathbb{T}} | \mathbf{k}_A\mathbf{k}_B \rangle =_{\mathcal{O}(\lambda)} {}_0\langle \mathbf{p}_1\mathbf{p}_2 | \mathcal{T}[\frac{-i\lambda}{4!} \int d^4x \, \hat{\phi}_I^4(\mathbf{x})] | \mathbf{k}_A\mathbf{k}_B \rangle_0 \tag{296}$$

$$= {}_0\langle \mathbf{p}_1\mathbf{p}_2 | \mathcal{N}[\frac{-i\lambda}{4!} \int d^4x \, \hat{\phi}_I^4(\mathbf{x}) + \underset{\text{contractions}}{\overset{\text{all}}{}}] | \mathbf{k}_A\mathbf{k}_B \rangle_0 \tag{297}$$

To see what kinds of terms that survive to order λ, consider the interacting creation field operator interacting with the free four-momentum eigenstate

$$\hat{\phi}_I^\dagger(\mathbf{x}) | \mathbf{p} \rangle_0 = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} e^{-i\mathbf{k}\cdot\mathbf{x}} \hat{a}_k | p \rangle \tag{298}$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} e^{-i\mathbf{k}\cdot\mathbf{x}} \hat{a}_k \cdot \sqrt{2\omega_p} \hat{a}_p^\dagger | 0 \rangle \tag{299}$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} e^{-i\mathbf{k}\cdot\mathbf{x}} \cdot (2\pi)^3 \delta^{(3)}(k - p)\sqrt{2\omega_p} | 0 \rangle \tag{300}$$

$$\hat{\phi}_I^\dagger(\mathbf{x}) | \mathbf{p} \rangle_0 = e^{-ip\cdot x} | 0 \rangle. \tag{301}$$

To deal with momentum eigenstates, such as $|\mathbf{k}_A\mathbf{k}_B\rangle$, include them in *extended contractions* by defining

$$\overparen{\hat{\phi}_I^\dagger(\mathbf{x}) | p \rangle} = e^{-i\mathbf{p}\cdot\mathbf{x}} \tag{302}$$

$$\overparen{\langle p | \hat{\phi}_I^\dagger(\mathbf{x})} = e^{i\mathbf{p}\cdot\mathbf{x}} \tag{303}$$

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Now, we can prove (**Exercise**) an extended version of Wick's theorem, where the time-ordered product of field operators in the presence of incoming and outgoing momentum eigenstates is equal to the sum over all possible contractions, including contractions of the field operators with the states as well.

$$
{}_0 \langle \mathbf{p}_1 \ldots \mathbf{p}_n | \, \mathcal{T}[\text{field operators}] \, | \mathbf{k}_1 \ldots \mathbf{k}_n \rangle_0 = \sum {}^{\text{all possible full contractions}}_{\text{including momentum eigenstates}} \quad (304)
$$

This allows us to approximate the $\mathcal{S}$-matrix transition amplitudes, justified up to the assumption about momentum eigenstates in the free theory being related to momentum eigenstates in the interacting theory.

13 Lecture 13: Feynman Diagram Expansions in φ^4 Theory

In overview, thus far we have take a progression of reasonable, small steps to building a relativistic quantum field theory. We established Lorentz invariance as a symmetry of the theory, which may not quite be a fundamental symmetry of nature, but has not been violated by any experiments to date. Next, we established a unitary representation of the Poincaré group (difficult since the Poincaré group is not compact, unlike rotation groups), where space and time can transform into other via Lorentz boosts, meaning that energy and momentum can also exchange roles under Poincaré symmetries. This also causes difficulties in writing down quantum mechanical Hamiltonians that behave appropriately, but, fortunately, we can invoke locality in the theory, which is also seemingly fundamental to nature. Motivated by classical examples, such as the Klein-Gordon equation, a free theory, we write the relativistic quantum field theory, namely the quantum Klein-Gordon field, by "just putting hats" on the operators, and it worked.

To account for interactions, we added a perturbative φ^4 term to the quantum Klein-Gordon Hamiltonian

$$\hat{H}_{KG} \to \hat{H}_{KG} + \hat{H}_{\varphi^4} \tag{305}$$

Where $\hat{H}_{\varphi^4}$ is Lorentz invariant, but is also an unbounded operator on any Hilbert space, such that $||\hat{H}_{\varphi^4}|| \to \infty$. Though this is not mathematically rigorous, our forefathers and foremothers have shown through physical rigor, as well as time and time again by using tools outside of the realm of their mathematical applicability, that this is an acceptable perturbative term to account for field interactions.

Examples of Mathematically Non-Rigorous Applications

- The building block of perturbation theory assumes that the interacting theory vacuum state is proportional to the free theory vacuum state via the time-evolution operator of the interacting theory

    $|\Omega\rangle \propto \lim_{t\to\infty} e^{-i\hat{H}_{\varphi^4}t} |\Omega_0\rangle$.

    - Even for finite dimensional systems, this limit does not exist and is oscillatory. The limit can make sense for Hamiltonians with a continuous spectra with a few particles.

- The momentum eigenstates require preparation of delta functions in the momenta, and are assumed to be related similar to the vacuum states

    $|\mathbf{p}_1 \ldots \mathbf{p}_n\rangle = \lim_{t\to\infty} e^{-i\hat{H}_{\varphi^4}t} |\mathbf{p}_1 \ldots \mathbf{p}_n\rangle_0$.

- The Dyson series, a Taylor expansion of the interacting theory time evolution operator can not be expected to converge

$$e^{-i\hat{H}_{\varphi^4}t} = e^{-i\hat{H}_{KG}t} - i \int_0^t dt'\, e^{-i\hat{H}_{KG}t'} \hat{H}_I e^{i\hat{H}_{KG}t'} + \dots .$$

We are often punished by infinities, where we would prefer finite numbers. Some infinities are trivially removable with no operational consequence, such as the ground state energy shift in the Klein-Gordon theory, such that $E_0 \to E_0 + \infty$. Other infinities can be factored away by rescaling observables, such as in the case of vacuum bubbles (e.g., $\frac{\infty}{\infty} = 1$). Yet other infinities just don't go away, such as in the diagram of two internal vertices.

Perhaps the choice of interaction terms is the cause of all of these infinities, and we can impose a cutoff to the theory that will eliminate some infinities. For example, in the φ^4 theory

$$\hat{H}_{\varphi^4} \to \hat{H}_{\varphi^4}(\Lambda) \tag{306}$$

Where Λ is a cutoff up to some family of models/theories, and the norm of the cutoff Hamiltonian is proportional to the cutoff, which may be large, but not infinity, such that $||\hat{H}_{\varphi^4}(\Lambda)|| \propto \Lambda$.

For example, consider the quantum harmonic oscillator cutoff, where $||\hat{H}(N)|| = N$

$$\hat{H} = \sum_{n=0}^{\infty} (n+\frac{1}{2}) |n\rangle \langle n| \to \hat{H}(N) = \sum_{n=0}^{N} (n+\frac{1}{2}) |n\rangle \langle n| + \sum_{n>N} (N+\frac{1}{2}) |n\rangle \langle n| \tag{307}$$

The major concern with imposing cutoffs is whether observables depend on the cutoff or not. To eschew this issue, we allow unobservable parameters, coupling constants, to shift and absorb all the cutoff dependence.

Renormalization

This practice is called **renormalization**, and it works. We make the hypothesis that the Hamiltonian depends on some number of parameters, called coupling constants, such that $\hat{H} = \hat{H}(z_1, \dots, z_n)$. The mapping from the coupling constants to observables is not expected to be, and is usually not, bijective. For example,

$$\hat{H}_{\varphi^4}(\lambda, m) = \int d^3x \left(\nabla^2 \hat{\phi} + m^2 \hat{\phi}^2 + \frac{\lambda}{4!} \hat{\phi}^4 \right) \tag{308}$$

This Hamiltonian corresponds to a list of observables $\mathcal{O}_j(z_1, \dots, z_n)$, $j = 1, 2, \dots$, with complicated (nonlinear) dependencies with the coupling constants, which usuaslly exist on a smooth manifold before mapping to observables.

For example, consider the spectral gap $\mathcal{O} = E_1 - E_0$, where $\mathcal{O} = \mathcal{O}(z_1 + c, z_2, \dots, z_n)$, and $\hat{H} = z_1 \cdot \mathbb{I} + \hat{H}'(z_2, \dots, z_n)$.

Add the cutoff dependence to the observables, such that

$$\hat{H}_\Lambda(z_1, \ldots, z_n) \to \mathcal{O}_{expt.} = \mathcal{O}_j(z_1, \ldots, z_n; \Lambda) \tag{309}$$

If "lucky", each of the coupling constants can absorb all of the Λ-dependence, and none of the observables will depend directly on the cutoff

$$\mathcal{O}_{expt.} = \mathcal{O}_j(z_1, \ldots, z_n; \Lambda) = \mathcal{O}_j(z_1(\Lambda), \ldots, z_n(\Lambda)). \tag{310}$$

If the above is true, then the theory is renormalizable, and is now defined by a highly overdetermined set of equations, such that there are a finite number of parameters z_i and an infinite list of equations (observables) $\mathcal{O}_j$ to solve. Renormalizable theories effectively have no cutoff, since the remaining infinities are eliminated by changing the order of limits.

1. Take the limit as the cutoff tends to infinity, $\Lambda \to \infty$, then compute observable quantities.

2. Vice versa.

Typical Terms in Scattering Experiments

Scattering experiments generate interaction terms such as the following

$$_0 \langle p_1 p_2 \ldots p_n | \, \mathcal{T} \, [\frac{\lambda}{4!} \int d^4x \, \hat{\phi}_I^4(x)] \, |q_A q_B\rangle_0 \,. \tag{311}$$

Apply (generalized) Wick's theorem to calculate the contractions, and sum over three types of terms (to order λ) encountered. Note that $\hat{\phi}_I(x) = \hat{\phi}$ in the following. Wick's theorem applied to $\mathcal{T}[\hat{\phi}^4(x)]$ yields a sum over the normal ordering of all of the following contractions, such that

$$\mathcal{T}[\hat{\phi}^4(x)] = \text{fully contracted} + \text{partially contracted} + \text{uncontracted} \tag{312}$$

$$\textbf{Fully contracted}: \ \mathcal{N}[\overset{\sqcap \ \sqcap}{\hat{\phi}\hat{\phi}\hat{\phi}\hat{\phi}}] + \text{other fully contracted}$$

$$\textbf{Partially contracted}: \ \mathcal{N}[\overset{\sqcap}{\hat{\phi}\hat{\phi}}\hat{\phi}\hat{\phi} + \hat{\phi}\hat{\phi}\overset{\frown}{\hat{\phi}\hat{\phi}}] + \text{other partially contracted}$$

$$\textbf{Uncontracted}: \ \mathcal{N}[\hat{\phi}\hat{\phi}\hat{\phi}\hat{\phi}].$$

The diagrammatic contributions to the integrals from each type of contracted term to the $\mathcal{S}$-matrix have the following forms (for the $n = 2$ case)

Type 1, fully contracted:

$$\frac{-i\lambda}{4!} \int d^4x \, _0 \langle p_1 p_2 | \, \overset{\sqcap \ \sqcap}{\hat{\phi}\hat{\phi}\hat{\phi}\hat{\phi}} \, |q_A q_B\rangle_0 = \tag{313}$$
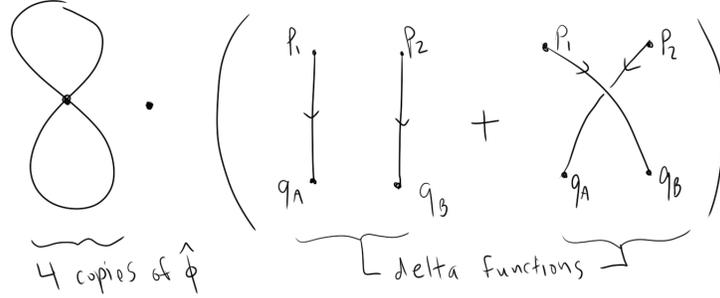
Figure 24: Diagrammatic contributions of fully contracted terms: a product of fully connected diagrams, from four copies of the field operator (figure eight), and delta functions imposing a sort of momentum conservation.

The full contraction is just a $\mathbb{C}$ number, making the integral over the inner product of momentum eigenstates.

Type 2, partially contracted:

$$\frac{-i\lambda}{4!} \int d^4x \; {}_0\langle p_1 p_2| \overset{\sqcap}{\hat{\phi}\hat{\phi}} \mathcal{N}[\hat{\phi}\hat{\phi}] |q_A q_B\rangle_0 = \frac{-i\lambda}{4!} \int d^4x \, (\,(1) + (2) + (3)\,) \quad (314)$$

The partial contractions have three more types of normal orderings, as labelled above: $(1) + (2) + (3)$

1. **Right:** $\hat{\phi}\hat{\phi} \; {}_0\langle p_1 p_2| \hat{\phi}\hat{\phi} |q_A q_B\rangle_0 +$ all other contractions to the right.

2. **Left:** $\hat{\phi}\hat{\phi} \; {}_0\langle p_1 p_2| \hat{\phi}\hat{\phi} |q_A q_B\rangle_0 +$ all other contractions to the left.

3. **Both:** $\hat{\phi}\hat{\phi} \; {}_0\langle p_1 p_2| \hat{\phi}\hat{\phi} |q_A q_B\rangle_0 +$ all other contractions one left and one right.

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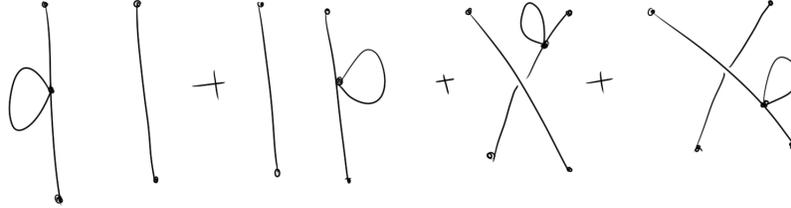
Figure 25: Diagrammatic contributions of partially contracted terms.

Note that the contraction of the interacting field operators and the momentum eigenstates, as in the three cases above, contribute incoming/outgoing momenta, a vertex, and three external legs.
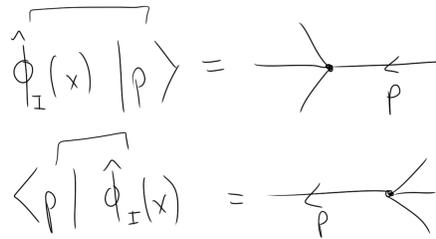

Figure 26: Diagrammatic contributions of contractions between intertacting field operators and momentum eigenstates.

Type 3, uncontracted:

$$\frac{-i\lambda}{4!} \int d^4x \; {}_0\langle p_1 p_2 | \mathcal{N}[\hat{\phi}\hat{\phi}\hat{\phi}\hat{\phi}] |q_A q_B\rangle_0 = \frac{-i\lambda}{4!} \cdot 4! \cdot \int d^4x\, e^{-i(q_A + q_B - p_1 - p_2)\cdot x}$$
$$= -i\lambda (2\pi)^4 \delta^{(4)}(q_A + q_B - p_1 - p_2)$$
$$= \text{Diagram below.}$$

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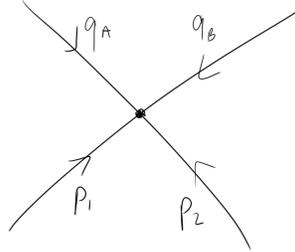
Figure 27: Uncontracted term contributes diagram that enforces momentum conservation.

Adding all of the terms together in the full expansion of the interacting component of the $\mathcal{S}$-matrix, namely $_0 \langle p_1 p_2 | i\hat{\mathbb{T}} | q_A q_B \rangle_0$, we get a series lke the following

Figure 28: Full diagrammatic expansion of the interacting component fo the $\mathcal{S}$-matrix.

These are the three types of diagrams, with respect to connectedness, that are always encountered: fully connected, partially connected, and vacuum bubble times cully connected.

Figure 29: Three types of connectedness encountered in diagrammatic expansions of the interacting component of the $\mathcal{S}$-matrix.

Vacuum Bubbles and Fully Connected Diagrams

Claim, here without proof, that products of vacuum bubbles and fully connected diagrams exponentiate, and, further, the only diagrams which contribute to the $\mathcal{S}$-matrix are the fully connected diagrams. There still exist fully connected diagrams (integrals) that result in infinities. To quell these infinities, either impose a cutoff scale, **or** argue that such a diagram does not contribute to the $\mathcal{S}$-matrix.
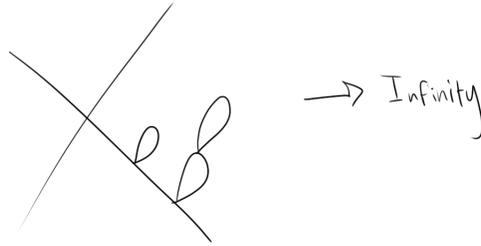
Figure 30: An example of a fully connected diagram that results in infinity.

External leg corrections may be used to "amputate" parts of a diagram that will remove infinities from the external legs. This process represents a projection from the free momentum eigenstates to the interacting momentum eigenstates, just as in the projection of the vacuum state $|\Omega\rangle$. Thus, external leg corrections factorize the diagram.

$$|q_B\rangle_0 \to |q_B\rangle \tag{315}$$
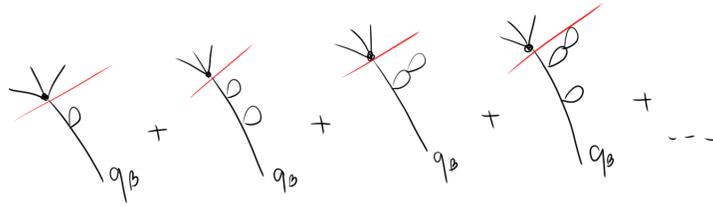
Figure 31: Single cuts made from the body of the diagram are called *external leg corrections*, and "amputate" off infinities.

Amputation can not remove all infinities, since it is defined via the external leg corrections. Self-interactions on internal legs are not removable via amputation.
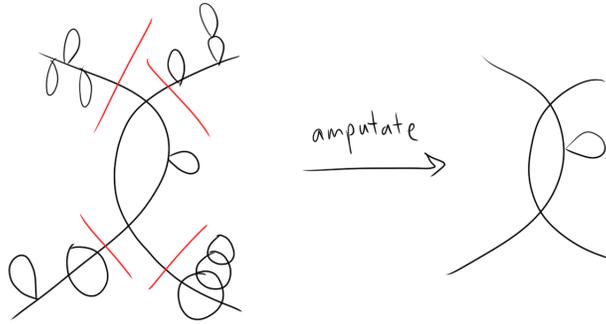
Figure 32: An example of an amputation process where not all infinities can be removed, since it is not on an external leg.

And the final statement of the Feynman rule for the φ^4 interaction theory reads

$$i\mathcal{M}(2\pi)^4\delta^{(4)}(q_A + q_B - \sum_f p_f) = \begin{pmatrix} \text{sum of all connected and} \\ \text{amputated Feynamn diagrams} \\ \text{with incoming momenta } q_A,\, q_B \\ \text{and outgoing momenta } p_f \end{pmatrix} \quad (316)$$

14 Lecture 14: The Dirac Field

We've built a relativistic quantum field theory for the (complex) scalar field, or spinless particles, around the Klein-Gordon equation and its Lorentz group invariant, conserved quantities, or the generators of Lorentz transformations (e.g., boosts and rotations), which obey the Lie algebra bracket

$$[J^{\rho\sigma}, J^{\tau\nu}] = \eta^{\sigma\tau} J^{\rho\nu} - \eta^{\rho\tau} J^{\sigma\nu} + \eta^{\rho\nu} J^{\sigma\tau} - \eta^{\sigma\nu} J^{\rho\tau}. \tag{317}$$

We now build a relativistic quantum field theory for fermions, particles of spin-$\frac{1}{2}$, which requires a (complex) vector, or spinor, field to describe the dynamics of the fermionic field. Our strategy is similar to the scalar field case:

- Build a relativistic classical field theory.

- Quantize the field (by guess).

- Check for the (projective) representation of the Lorentz group.

To begin with building the relativistic classical field theory, given a set of objects obeying the Lie bracket above, we construct a Lagrangian by first constructing the representation of the Lorentz group via $g = e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}}$, where $\omega_{\mu\nu}$ are the Lorentz boost parameters, and then calculating Lagrangian densities invariant under the representation g.

Dirac's Motivation

Dirac desired to factorize the Klein-Gordon equation, and craft an equation of motion linear in spacetime derivatives $\partial_\mu \equiv \partial/\partial x^\mu$, the square root of the Klein-Gordon equation, such that any solution to the Dirac equation is also a solution to the Klein-Gordon equation, but not necessarily vice versa. Linearity in the spacetime derivatives ensures that the associated probability current is positive definite, and the positive energy solutions may be separated from the negative energy solutions inherent to solutions of the Klein-Gordon equation.

The Dirac equation, which we will motivate and derive, reads

$$(i\gamma^\mu \partial_\mu - m)\psi = 0. \tag{318}$$

Where the coefficients of the spacetime derivatives $i\gamma^\mu$ can not be just four scalar quantities, as the vector of coefficients would then define a direction, and the Dirac equation would not be Lorentz invariant. The γ^μ must actually be 4×4 matrices, and the field ψ must be a four-vector.

To recover the Klein-Gordon equation from the Dirac equation, multiply the Dirac equation by $(i\gamma^\nu \partial_\nu + m)$, and compare to the Klein-Gordon equation $(\partial^2 + m^2)\psi = 0$ to see that the "squared" Dirac equation becomes the Klein-Gordon equation if the *gamma matrices* obey the following anticommutation relation

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \cdot \mathbb{I}_{n\times n}. \tag{319}$$

Where $\eta^{\mu\nu}$ is the Minkowski metric tensor.

Generators of Fermionic Field (Lorentz) Transformations

To calculate the generators of the Lorentz transformations of the fermionic field (spin-$\frac{1}{2}$ particles), suppose that we have that set of 4 $n \times n$ matrices γ^μ that satisfy the anticommutation relation $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \cdot \mathbb{I}_{n\times n}$. Obtain a solution in terms of the gamma matrices that obeys the Lie algebra bracket above

$$J^{\mu\nu} \to S^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu, \gamma^\nu]. \tag{320}$$

It is an **(exercise)** to check that these generators obey Lie algebra bracket above for the Lorentz transformation, and that four dimensions is the minimum of the gamma matrices dimensionality for nontrivial solutions.

The gamma matrices that correctly form the Lorentz group in (3+1)d spacetime were crafted by Dirac via the Pauli spin (sigma) matrices

$$\gamma^0 = \left(\begin{array}{c|c} 0 & \mathbb{I} \\ \hline \mathbb{I} & 0 \end{array} \right) \quad \gamma^j = \left(\begin{array}{c|c} 0 & \sigma^j \\ \hline -\sigma^j & 0 \end{array} \right). \tag{321}$$

Where the Pauli sigma matrices read

$$\sigma^1 = \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \quad \sigma^2 = \left(\begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right) \quad \sigma^3 = \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right). \tag{322}$$

The timelike components S^{0j} yield three ($j = 1, 2, 3$) boost generators of the Lorentz group (of the Lie algebra above)

$$S^{0j} = \frac{i}{4}\left[\gamma^0, \gamma^j\right] = -\frac{i}{2}\left(\begin{array}{cc} \sigma^j & 0 \\ 0 & -\sigma^j \end{array} \right). \tag{323}$$

The other six generators from the spacelike components have the form

$$S^{jk} = \frac{i}{4}\left[\gamma^j, \gamma^k\right] = \frac{1}{2}\epsilon^{jkl}\left(\begin{array}{cc} \sigma^l & 0 \\ 0 & -\sigma^l \end{array} \right) \tag{324}$$

Where the sigma matrices obey the commutation relation: $[\sigma^j, \sigma^k] = 2i\epsilon^{jkl}\sigma^l$.

Construction of Gamma Matrices

To build the operators that obey the desired anticommutation relation, and also form a *Clifford algebra*, implement the **Jordan-Wigner transformation**, which maps the Pauli sigma matrices to fermionic creation and annihilation operators.

Jordan-Wigner Transformation

Suppose that you want to represent operators that obey the (fermionic) anti-commutation relation

$$\{\mu^a, (\mu^b)^\dagger\} = \delta^{ab}, \quad a, b = 1, 2, \ldots, n. \tag{325}$$

Consider the operator $\sigma^+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and the construction of successive operators which all operate on the same n-dimensional space, permissing the expanse of n direct products with the unit operator

$$\mu^1 = \sigma^+ \otimes \mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I}$$
$$\mu^2 = \sigma^3 \otimes \sigma^+ \otimes \mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I}$$
$$\mu^3 = \sigma^3 \otimes \sigma^3 \otimes \sigma^+ \otimes \mathbb{I} \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I}$$
$$\mu^4 = etc. \ldots$$

These μ^j operators obey the fermioic anticommutation relations, since $\{\sigma^j, \sigma^k\} = 2\delta^{jk}$, and form a matrix representation of the anticommutation relation given in the Jordan-Wigner transformation.

An inefficient method to construct a representation of the gamma matrices, though reducible to the above μ^j matrices, which honors the anticommutation relation, starts with the composition of the timelike gamma matrix

$$\gamma^0 = \sigma^1 \otimes \mathbb{I}. \tag{326}$$

Next, we know that σ^2 anticommutes with σ^1 via $\{\sigma^j, \sigma^k\} = 2\delta^{jk}$, and thus we construct the following, noting that to satisfy $(\gamma^j)^2 < 0$, scalar multiplication by the imaginary unit i is required

$$\gamma^j \equiv i\sigma^2 \otimes \sigma^j. \tag{327}$$

The Dirac Spinor

The set of matrices γ^μ satisfying the anticommutation relation $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \cdot \mathbb{I}_{n \times n}$ are called the **Dirac matrices**, and the four-component object that transforms correctly under the Lorentz group is called the **Dirac spinor**

$$\psi \to e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}}\psi \tag{328}$$

This is the definition of the spinor in different reference frames, and defines the **spinor field** as a mapping

$$\psi : \mathbb{R}^{1,3} \to \mathbb{C}^4. \tag{329}$$

Transforming as

$$\psi_a(x) = \sum_{b=0}^{3} [\Lambda_{1/2}]_{ab} \ \psi_b(\Lambda^{-1}x), \ \Lambda \in SO(3) \tag{330}$$

Where Λ is the usual representation for the Lorentz transformation, and $\Lambda_{1/2}$ has the same representation structure and obeys the same Lie algebra of the Lorentz transformation, but different elements: the 4×4 matrix Lie algebra bracket $S^{\mu\nu}$ built out of the gamma matrices

$$\Lambda = e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}} \tag{331}$$

$$\Lambda_{1/2} = e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}} \tag{332}$$

Every representation of the Lorentz group can be used to build Lorentz invariant field equations, such that $\mathcal{D}\psi = 0$. To build the field equations for $\psi_a(x)$, first guess and check (**Exercise**) that the Klein-Gordon equation is satisfied

$$(\partial^2 + m^2)\psi_a(x) = 0, \; \forall \, a \tag{333}$$

The next guess, by Dirac, is to find a satisfactory field equation that is linear in spacetime derivatives. This requires the following identity as an auxiliary computation between the gamma matrices and the two generators, first confirmed by Dirac (**Exercise**)

$$[\gamma^\mu, S^{\rho\sigma}] = (J^{\rho\sigma})^\mu{}_\nu \gamma^\nu \tag{334}$$

This is reminiscient of how the four-vector transforms under the Lorentz transformation, and we motivatedly write, to first order in ω

$$(1 + \frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma})\gamma^\mu(1 - \frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}) = (1 - \frac{i}{2}\omega_{\rho\sigma}J^{\rho\sigma})^\mu{}_\nu\gamma^\nu \tag{335}$$

$$e^{\frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}}\gamma^\mu e^{-\frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}} \approx (e^{-\frac{i}{2}\omega_{\rho\sigma}J^{\rho\sigma}})^\mu{}_\nu\gamma^\nu \tag{336}$$

$$\Lambda_{1/2}^{-1}\gamma^\mu\Lambda_{1/2} = \Lambda^\mu{}_\nu\gamma^\nu \tag{337}$$

This shows that the gamma matrices transform exactly like a four-vector under the Lorentz transformation. Contracted against another vector, that transforms accordinngly, will produce another Lorentz invariant object. The differential operator ∂_μ transforms as such, making $\gamma^\mu\partial_\mu$ Lorentz invariant.

Thus, the Lorentz invariant field equation, the Dirac equation, reads

$$(i\gamma^\mu\partial_\mu - m)\psi = 0 \tag{338}$$

Check the invariance of the Dirac equation by substituting the following quantities for the spinor and spacetime derivative into the Dirac equation (**Exercise**)

$$\psi(x) \to \Lambda_{1/2}\psi\left(\Lambda^{-1}x\right) \tag{339}$$

$$\partial_\mu \to (\Lambda^{-1})^\nu{}_\mu\partial_\nu. \tag{340}$$

Making these subsitutions, the Dirac equation becomes

$$\left(i\gamma^\mu(\Lambda^{-1})^\nu{}_\mu\partial_\nu - m\right)\Lambda_{1/2}\,\psi(\Lambda^{-1}x) = 0 \tag{341}$$

$$\Lambda_{1/2}\Lambda_{1/2}^{-1}\cdot\left(i\gamma^\mu(\Lambda^{-1})^\nu{}_\mu\partial_\nu - m\right)\Lambda_{1/2}\,\psi(\Lambda^{-1}x) = 0 \tag{342}$$

$$\Lambda_{1/2}\left(i\Lambda_{1/2}^{-1}\gamma^\mu\Lambda_{1/2}\,(\Lambda^{-1})^\nu{}_\mu\partial_\nu - m\right)\psi(\Lambda^{-1}x) = 0 \tag{343}$$

$$\Lambda_{1/2}\left(i\Lambda^\mu{}_\sigma\gamma^\sigma(\Lambda^{-1})^\nu{}_\mu\partial_\nu - m\right)\psi(\Lambda^{-1}x) = 0 \tag{344}$$

$$\Lambda_{1/2}\left(i\gamma^\nu\partial_\nu - m\right)\psi(\Lambda^{-1}x) = 0 \tag{345}$$

Going from line 2 to 3 utilizes the fact that $(\Lambda^{-1})^\nu{}_\mu\partial_\nu$ is a linear operator, meaning the quantity $\Lambda_{1/2}^{-1}\gamma^\mu\Lambda_{1/2}$ also transforms like a four-vector and is Lorentz invariant. Going from line 4 to 5 uses the fact that $(\Lambda^{-1})^\nu{}_\mu\partial_\nu\,\psi(\Lambda^{-1}x)$ is a delta function. The left hand side becomes zero, and the Dirac equation is Lorentz invariant.

The Dirac equation contains the Klein-Gordon equation

$$(i\gamma^\mu\partial_\mu - m)\psi = 0$$

$$(-i\gamma^\nu\partial_\nu - m)\cdot(i\gamma^\mu\partial_\mu - m)\psi = 0$$

$$(\frac{1}{2}\{\gamma^\mu,\gamma^\nu\}\partial_\mu\partial_\nu + m^2)\psi = 0$$

$$(\partial^\mu\partial^\nu + m^2)\psi = 0$$

$$(\Box^2 + m^2)\psi = 0$$

The Dirac equation is first order in all four spacetime coordinates, and is thus a stronger condition imposed on the components of the field ψ.

15 Lecture 15: The Dirac Equation and its Solutions

The Dirac equation $(i\gamma^\mu \partial_\mu - m)\,\psi = 0$ is motivated for us by the question of the dynamics of fermionic fields, espeically electrons. We introduce a new representation of the Lorentz group with intrinsic angular momentum by producing the gamma matrices γ^μ, such that the anticommutation relation is satisfied.

$$\{\gamma^\mu,\,\gamma^\nu\} = \eta^{\mu\nu} \cdot \mathbb{I} \tag{346}$$

And recall that the gamma matrices contain the Pauli spin matrices and have the form

$$\gamma^0 = \left(\begin{array}{c|c} 0 & \mathbb{I} \\ \hline \mathbb{I} & 0 \end{array}\right) \quad \gamma^j = \left(\begin{array}{c|c} 0 & \sigma^j \\ \hline -\sigma^j & 0 \end{array}\right). \tag{347}$$

This will yield a new Lorentz-invariant equation of motion, and the steps to get there include

1. Calculating the Hamiltonian denisty from the Lagrangian density and guessing a quantization.

2. Entering the zero-mass limit, where the equation of motion decouples.

3. Solve the wave equation in the free particle basis.

Recall that the Dirac spinor Lorentz-transforms according to

$$\psi(x) \to \Lambda_{1/2}\psi(\Lambda^{-1}x) \tag{348}$$

To build the Lagrangian density, we need to define some Lorentz scalars. A first guess may be to (incorrectly) define the Lorentz scalar as the quantity $\psi^\dagger(x)\psi(x)$, which Lorentz transforms as

$$\psi^\dagger(x)\psi(x) \to \psi^\dagger \Lambda_{1/2}^\dagger \Lambda_{1/2}\psi \tag{349}$$

The issue with this definition is that the representation of the Lorentz group can be unitary if it is finite dimensional, and, in general, $\Lambda_{1/2}^\dagger \Lambda_{1/2} \neq \mathbb{I}$, and $\Lambda_{1/2}$ is not unitary. The proper definition of the Lorentz scalar includes a gamma matrix factor to ensure unitarity.

Define the Lorentz scalar as $\psi^\dagger(x)\gamma^0\psi(x)$, and define the quantity $\bar{\psi}(x) = \psi^\dagger(x)\gamma^0$ to assist in proving that $\bar{\psi}\psi$ is indeed a Lorentz scalar.

Exercise: Prove this using the

Lemma: $\Lambda_{1/2}^\dagger \gamma^0 = \gamma^0 \Lambda_{1/2}^{-1}$.

Similarly, define the Lorentz vector $v^\mu = \bar{\psi}\gamma^\mu\psi$.

Therefore, the Lagrangian density for the Dirac field is

$$\mathcal{L}_{Dirac} = \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi \tag{350}$$

Exercise: Prove that $\mathcal{L}_{Dirac}$ is Lorentz scalar.
Exercise: Apply the Euler-Lagrange equation to obtain the Dirac equation.

Dirac Field Hamiltonian

The conjugate momentum to the Dirac spinor ψ is the quantity $i\psi^\dagger$, yielding the Hamiltonian

$$H = \int d^3x \; \bar{\psi}\left(-i\underline{\gamma}\cdot\underline{\nabla} + m\right)\psi \tag{351}$$

$$H = \int d^3x \; \psi^\dagger\left(-i\gamma^0\underline{\gamma}\cdot\underline{\nabla} + m\gamma^0\right)\psi \tag{352}$$

Where the underlined vectors are only the spatial components, such that

$$\underline{\gamma}\cdot\underline{\nabla} = \sum_{j=1}^{3}\gamma^j\partial_j.$$

And we have the dynamics of a classical Dirac field.

Zero-mass Limit

Let's see how the Dirac equation and the four-component Dirac spinor decouples in the zero-mass limit to the two-component *Weyl spinors* $\psi_{L/R}(x) \in \mathbb{C}^2$.

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix} \rightarrow \begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix} \tag{353}$$

Under the Lorentz transformation, Weyl spinors transform to themselves

$$\psi_L \rightarrow (\mathbb{I} - i\underline{\theta}\cdot\frac{1}{2}\underline{\sigma} - \underline{\beta}\cdot\frac{1}{2}\underline{\sigma})\psi_L \tag{354}$$

$$\psi_R \rightarrow (\mathbb{I} - i\underline{\theta}\cdot\frac{1}{2}\underline{\sigma} + \underline{\beta}\cdot\frac{1}{2}\underline{\sigma})\psi_R \tag{355}$$

Where the rotation and boost three-vectors are, respectively, $\underline{\theta} = (\theta_1, \theta_2, \theta_3)$ and $\underline{\beta} = (\beta_1, \beta_2, \beta_3)$.

The left-handed spinor ψ_L contains the right-handed spinor ψ_R, seen by taking the complex conjugate and multiplying by σ^2, since this causes a *spin flip*, such that multiplication by σ^2 give the antipode on the Bloch sphere.

$$\sigma^2 \underline{\sigma}^* = -\underline{\sigma}\sigma^2 \tag{356}$$

Therefore, the quantity $\sigma^2\psi_L^*$ transforms like ψ_R.

The coupling of the Weyl spinors is represented by the matrix equation (**Exercise**)

$$(i\gamma^\mu \partial_\mu - m)\,\psi = \begin{pmatrix} -m & i(\partial_0 + \underline{\sigma}\cdot\underline{\nabla}) \\ i(\partial_0 - \underline{\sigma}\cdot\underline{\nabla}) & m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0 \tag{357}$$

Now, suppose that $m = 0$, such that ψ_L and ψ_R become two independent solutions to the Dirac equation, and we obtain a represntation of the Poincaré group. The equations of motion in the zero-mass limit are then

$$i(\partial_0 - \underline{\sigma}\cdot\underline{\nabla})\psi_L = 0 \tag{358}$$
$$i(\partial_0 + \underline{\sigma}\cdot\underline{\nabla})\psi_R = 0 \tag{359}$$

Define the notation

$$\sigma^\mu = (1, \underline{\sigma}) \tag{360}$$
$$\bar{\sigma}^\mu = (1, -\underline{\sigma}) \tag{361}$$

Then the gamma matrices become

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \tag{362}$$

And we rewrite the Dirac equations of motion as

$$i(\bar{\sigma}\cdot\partial)\psi_L = 0 \tag{363}$$
$$i(\sigma\cdot\partial)\psi_R = 0 \tag{364}$$

Free Particle Solutions to the Dirac Equation

The Dirac equation is a linear system of partial differental equations (PDEs), and the solutions may be expressed in terms of plane waves

$$\psi(x) = e^{-ip\cdot x}u(p) \tag{365}$$

Where $u(p) \in \mathbb{C}^4$ and are the zero eigenvectors of the Dirac matrix. Linear combinations of $\psi(x)$ yield a general solution, with the constraint of *on-shell*

momentum, such that $p^2 = m^2$, $p^0 > 0$.

Substitute the plane wave solution in to Dirac equation to obtain the matrix equation

$$(p_\mu \gamma^\mu - m \cdot \mathbb{I}) \, u(p) = 0 \tag{366}$$

Now, solving for $u(p)$, enter the rest frame, where $p_0 = (m, 0)$, and the matrix equation becomes

$$\left(m \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} - m \begin{pmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix} \right) \begin{pmatrix} u_L(p_0) \\ u_R(p_0) \end{pmatrix} = 0 \tag{367}$$

And solutions are of the form

$$u_L(p_0) = u_R(p_0) = \sqrt{m} \, \xi \tag{368}$$

Where ξ fixed, with normalization $\xi^\dagger \xi = 1$. Then the solution in the rest frame is

$$\psi^s(x) = e^{-ip \cdot x} u^s(p_0) = \sqrt{m} \begin{pmatrix} \xi^s \\ \xi^s \end{pmatrix} e^{-ip \cdot x} \tag{369}$$

Where $s = 1, 2$, and we may write $\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

The solution in a general reference frame is obtained via the spinor representation of the Lorentz boost Λ. To calculate the spinor representation of the Lorentz boost along any three-direction, recall that, for rapidity η

$$\Lambda = \cosh(\eta) \cdot \mathbb{I} + \sinh(\eta) \cdot \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{370}$$

$$= e^{-i \frac{1}{2} \omega_{\mu\nu} J^{\mu\nu}} \tag{371}$$

$$\Lambda = e^{-i \frac{1}{2} \omega_{03} J^{03}} \tag{372}$$

Where $J^{\mu\nu}$ obey the Lie algebra of the Lorentz group, $\omega_{\mu\nu}$ are the 6 boost parameters, and $\omega_{03} = 2\eta$ is the only nonzero term (**Exercise**).
To apply this transformation on the spinor, apply the matrix

$$\Lambda_{1/2} = e^{-\frac{i}{2} \omega_{03} S^{03}} \tag{373}$$

$$= e^{-\frac{i}{2} \cdot 2\eta \cdot \frac{i}{4} [\gamma^0, \gamma^3]} \tag{374}$$

$$= \begin{pmatrix} e^{-\frac{1}{2} \eta \sigma^3} & 0 \\ 0 & e^{\frac{1}{2} \eta \sigma^3} \end{pmatrix} \tag{375}$$

So, in the general reference frame, $p = \Lambda p_0$, the solutions have the form (**Exercise**)

$$\psi(x) = e^{-ip \cdot x} u^s(p = \Lambda p_0) \tag{376}$$

$$= \sqrt{m} \begin{pmatrix} e^{-\frac{1}{2}\eta \sigma^3} \xi^s \\ e^{\frac{1}{2}\eta \sigma^3} \xi^s \end{pmatrix} e^{-ip \cdot x} \tag{377}$$

$$= \begin{pmatrix} \sqrt{p \cdot \sigma} \xi^s \\ \sqrt{p \cdot \bar{\sigma}} \xi^s \end{pmatrix} e^{-ip \cdot x} \tag{378}$$

Where we used $(p \cdot \sigma)(p \cdot \bar{\sigma}) = m^2 = p^2$, and we have two linearly independent solutions to the Dirac equation, but we need four solutions per given momentum. The other two solutions come from the *plane wave ansatz* for the negative frequency solutions.

$$\psi_{negative}(x) = v(p) e^{ip \cdot x} \tag{379}$$

Similar to the positive frequency case, there are two linearly independent solutions

$$\psi_{negative}(x) = e^{ip \cdot x} v^s(p) \tag{380}$$

$$= \begin{pmatrix} \sqrt{p \cdot \sigma} \xi^s \\ -\sqrt{p \cdot \bar{\sigma}} \xi^s \end{pmatrix} e^{ip \cdot x} \tag{381}$$

To normalize these states, recall that $\bar{\psi}\psi$ is Lorentz invariant, with $\psi(x) = u^s(p) e^{-ip \cdot x}$, such that

$$\bar{\psi}\psi = \bar{u}^s(p) u^s(p) = 2m (\xi^s)^\dagger \xi^s \tag{382}$$

And more generally, for both positive and negative frequency solutions

$$\bar{\psi}\psi = \bar{u}^r(p) u^s(p) = 2m \delta^{rs} \tag{383}$$

$$\bar{\psi}_{negative} \psi_{negative} = \bar{v}^r(p) v^s(p) = -2m \delta^{rs} \tag{384}$$

And for mixed positive-negative frequency solutions, the normalization is zero

$$\bar{u}^r(p) v^s(p) = \bar{v}^r(p) u^s(p) = 0 \tag{385}$$

There we have four linearly independent solutions to the (classical) Dirac equation in a general reference frame.

16 Lecture 16: The Quantum Dirac Field

In the last lecture we found plane wave solutions to the classical Dirac equation. Now, we will guess a (quadractic) quantum theory that begins with the plane wave solutions of an effective classical theory in its large scale, low energy, high decoherence limit. The classical theory we make our guess has the Lagrangian density

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m \cdot \mathbb{I})\psi = \bar{\psi}(i\gamma^\mu\partial_\mu - m \cdot \mathbb{I})\psi \tag{386}$$

As a side note, the space of quantum theories is an abstract category constrained by unitary transformations (morphisms) and Hilbert space (objects).

First, a demonstration of how *not* to quantize the Dirac field.

Bosonic approach to quantization (the wrong way)

Suppose that we guess that the quantum Dirac field is a theory of many bosons, such that the equal-time commutation relation imposed on the field operators (note the hats), with spinor indices a, $b = \{1, 2, 3, 4\}$, is

$$[\hat{\psi}_a(x), \hat{\psi}_b(y)] = \delta^{(3)}(x-y)\delta_{ab}. \tag{387}$$

Use plane wave basis to expand the field operators and define creation/annihilation operators to act on momentum states, building a Fock space with basis

$$u^s(p)e^{ip\cdot x} \qquad v^s(p)e^{-ip\cdot x}. \tag{388}$$

The field operators are then Fourier transforms into coordinate space

$$\hat{\psi}(x) = \int \frac{d^3p}{(2\pi)^3} \ \frac{1}{\sqrt{2\omega_p}} e^{ip\cdot x} \sum_{s=1}^{2} \left(\hat{a}_p^s u^s(p) + (\hat{b}_{-p}^s)\dagger v^s(-p)\right). \tag{389}$$

Then the creation/annihilation operators are defined and they obey the commutation relations

$$[\hat{a}_p^r, (\hat{b}_q^s)^\dagger] = [\hat{b}_p^r, (\hat{b}_q^s)^\dagger] = (2\pi)^3\delta^{(3)}(p-q)\delta^{rs}. \tag{390}$$

And the quantized Hamiltonian is (**Exercise**)

$$\hat{H} = \int \frac{d^3p}{(2\pi)^3} \ \sum_{s=1}^{2} \omega_p \left((\hat{a}_p^s)^\dagger\hat{a}_p^s - (\hat{b}_p^s)^\dagger\hat{b}_p^s\right) + \infty \, const. \tag{391}$$

This Hamiltonian creates infinite bosons tor each infinitely lower energies and is not bounded below, and thus has no ground state; it is unstable.

Fermionic approach to quantization (the right way)

To correctly quantize the Dirac field, we impose equal-time *anticommutation* relations

$$\{\hat{\psi}_a(x), \hat{\psi}_b^\dagger(y)\} = \delta^{(3)}(x - y)\delta_{ab} \tag{392}$$

$$\{\hat{\psi}_a(x), \hat{\psi}_b(y)\} = 0 \tag{393}$$

$$\{\hat{\psi}_a^{\,\dagger}(x), \hat{\psi}_b^\dagger(y)\} = 0 \tag{394}$$

Again expand the field operators over momentum space via the Fourier transform

$$\hat{\psi}(x) = \int \frac{d^3p}{(2\pi)^3} \ \frac{1}{\sqrt{2\omega_p}} \sum_{s=1}^{2} \left(\hat{a}_p^s u^s(p) e^{-ip\cdot x} + (\hat{b}_p^s)^\dagger v^s(p) e^{ip\cdot x} \right) \tag{395}$$

$$\hat{\bar{\psi}}(x) = \int \frac{d^3p}{(2\pi)^3} \ \frac{1}{\sqrt{2\omega_p}} \sum_{s=1}^{2} \left(\hat{a}_p^s \bar{u}^s(p) e^{ip\cdot x} + (\hat{b}_p^s)^\dagger \bar{v}^s(p) e^{-ip\cdot x} \right). \tag{396}$$

So, the anticommutation relations then require that

$$\{\hat{a}_p^r, (\hat{a}_q^s)^\dagger\} = \{\hat{b}_p^r, (\hat{b}_q^s)^\dagger\} = (2\pi)^3 \delta^{(3)}(p - q)\delta^{rs}. \tag{397}$$

And all other brackets are zero.

The quantized Hamiltonian is

$$\hat{H} = \int \frac{d^3p}{(2\pi)^3} \ \sum_{s=1}^{2} \omega_p \left((\hat{a}_p^s)^\dagger \hat{a}_p^s + (\hat{b}_p^s)^\dagger \hat{b}_p^s \right) + \infty \, const. \tag{398}$$

Which is almost the same Hamiltonian as the bosonic approach, but note the additional minus sign in the acreation/annihilation operator quantity. This Hamiltonian is a positive operator, such that $\hat{H} \geq 0$ for all possible states, with a unique ground state $|\Omega\rangle$, such that

$$\hat{a}_p^s |\Omega\rangle = \hat{b}_p^s |\Omega\rangle = 0 \tag{399}$$

Therefore, this is the correct quantization of the Dirac field!

So far we have a Hilbert space and a time translation generator, (Hamiltonian) but to call this a *relativistic quantum field theory* we need all 10 operators to be Lorentz invariant: spatial translation (linear momentum), rotation (angular momentum), and boost operators.

Momentum operators of the Dirac field

The momentum operators, the generators of spatial translation, for the Dirac field are

$$\hat{p}_j = \int d^3x \ \hat{\psi}^\dagger(x)(-i\nabla_j)\hat{\psi}(x) \tag{400}$$

$$= \int \frac{d^3p}{(2\pi)^3} \sum_{s=1}^{2} \left(p_j \left((\hat{a}_p^s)^\dagger \hat{a}_p^s + (\hat{b}_p^s)^\dagger \hat{b}_p^s \right) \right) \tag{401}$$

Note that the operators and values $\hat{p}_j$, ∇_j, and p_j are 3-vectors, such that, for example, $\hat{p}_j = (\hat{p}_1, \hat{p}_2, \hat{p}_3) = (\hat{p}_x, \hat{p}_y, \hat{p}_z)$, and the j subscript denotes a single particle. The first line's integrand in the momentum density gotten by Noether's theorem.
(**Exercise**) Check that the required commutation relations hold, such that $[\hat{p}_j, \hat{H}] = 0$.

Now, the operators $(\hat{a}_p^s)^\dagger$ and $(\hat{b}_p^s)^\dagger$ create particles of energy ω_p and momentum p_j, suppressing the j subscript on the subscripted p, since (**Exercises**)

$$\hat{H}(\hat{a}_p^s)^\dagger |\Omega\rangle = \omega_p(\hat{a}_p^s)^\dagger |\Omega\rangle \tag{402}$$

$$\hat{p}_j(\hat{a}_p^s)^\dagger |\Omega\rangle = p_j(\hat{a}_p^s)^\dagger |\Omega\rangle \tag{403}$$

The sytematic, algorithmic way of obtaining the full unitary representation of the Poincaré group is to get the generators of the Lie algebra of the Lorentz group from Noether's theorem, put hats on them, and check the required commutation relations.

A less systematic way of obtaining the representation is done by introducing the "normalized" single-particle states

$$|p_j, s\rangle = \sqrt{2\omega_p}(\hat{a}_p^s)^\dagger |\Omega\rangle \tag{404}$$

With the "normalization" condition

$$\langle p_j, r | q_k, s \rangle = 2\omega_p(2\pi)^3 \delta^{(3)}(p_j - q_k)\delta^{rs}. \tag{405}$$

Then the unitary representation of the Lorentz group is defined via

$$U(\Lambda)\hat{a}_p^s U^\dagger(\Lambda) = \sqrt{\frac{\omega_{\Lambda p}}{\omega_p}} \hat{a}_{\Lambda p}^s \tag{406}$$

$$U(\Lambda)\hat{b}_p^s U^\dagger(\Lambda) = \sqrt{\frac{\omega_{\Lambda p}}{\omega_p}} \hat{b}_{\Lambda p}^s. \tag{407}$$

Note that since we are dealing on *equal-time* and the Lorentz transformation Λ acts on 4-vectors, the subscripted product Λp means to just the three spatial components from the transformation $\Lambda \cdot (\omega_p, p_x, p_y, p_z)$.

Now this gives us the full unitary representation of the Poincaré group since the Fock space is generated by products of the creation/annihilation operators on the vacuum state and the action is just extended to all of Fock space.

So we know how the Lorentz transformation acts on creation/annihilation operators. How does the Lorentz transformation act on the quantized Dirac (spinor) field operators?

$U(\Lambda)\hat{\psi}(x)U^\dagger(\Lambda)$

$$= U(\Lambda) \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \sum_{s=1}^{2} \left(\hat{a}_p^s u^s(p)e^{-ip\cdot x} + (\hat{b}_p^s)^\dagger v^s(p)e^{ip\cdot x} \right) U^\dagger(\Lambda)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} \sqrt{2\omega_p} \sum_{s=1}^{2} \left(\sqrt{\frac{\omega_{\Lambda p}}{\omega_p}} \left(\hat{a}_{\Lambda p}^s u^s(p)e^{-ip\cdot x} + (\hat{b}_p^s)^\dagger v^s(p)e^{ip\cdot x} \right) \right).$$

Recall that the quantity $\int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p}$ is a Lorentz invariant measure, such that we can change basis and freely apply Lorentz transformations to the variable of intergration; let $\widetilde{p} = \Lambda p$. Then (**Exercise**)

$U(\Lambda)\hat{\psi}(x)U^\dagger(\Lambda)$

$$= \int \frac{d^3\widetilde{p}}{(2\pi)^3} \frac{1}{2\omega_{\widetilde{p}}} \sqrt{2\omega_{\widetilde{p}}} \sum_{s=1}^{2} \left(\hat{a}_{\Lambda\widetilde{p}}^s u^s(\Lambda^{-1}\widetilde{p})e^{-i\widetilde{p}\cdot\Lambda^{-1}x} + (\hat{b}_{\widetilde{p}}^s)^\dagger v^s(\Lambda^{-1}\widetilde{p})e^{i\widetilde{p}\cdot\Lambda^{-1}x} \right)$$

$$= \Lambda_{1/2}\hat{\psi}(\Lambda^{-1}x).$$

So, we have taken an ad hoc definition of the creation/annihilation operators to construct the quantum field operators, and they Lorentz-transform according to the above; very nice!

Angular momentum operators of the Dirac field

The angular momentum operators, the generators of rotation, for the Dirac field classically transform by infinitesimal rotations

$$\psi(x) \rightarrow \Lambda_{1/2}\psi(\Lambda^{-1}x) \tag{408}$$

Where the infinitesimal rotation $\Lambda_{1/2}$ is

$$\Lambda_{1/2} \approx \mathbb{I} - \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu} = \mathbb{I} - \frac{i}{2}\theta \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} = \mathbb{I} - \frac{i}{2}\theta\Sigma^3 \tag{409}$$

The infinitesimal rotation of a spinor is calculated by first applying Taylor's theorem to first order

$$\delta\psi(x) = \psi'(x) - \psi(x) \tag{410}$$

$$= (\mathbb{I} - \frac{i}{2}\theta\Sigma^3)\psi(t, x + \theta y, y - \theta x, z) - \psi(x) \tag{411}$$

$$= -\theta(x\partial y - y\partial x + \frac{i}{2}\Sigma^3)\psi(x) \tag{412}$$

$$\equiv \theta\Delta\psi(x) \tag{413}$$

Now apply Noether's theorem, recalling that the time component of the conserved current due to rotation is

$$j^0 = \frac{\partial\mathcal{L}}{\partial(\partial_0\psi)}\Delta\psi = -i\bar{\psi}\gamma^0(x\partial y - y\partial x + \frac{i}{2}\Sigma^3)\psi. \tag{414}$$

By the "inverse" Noether's theorem, integrate the current over coordinate space to get the generators of rotations, the angular momentum operators, where we get an orbital contribution (wedge or corss product) and a spin contribution (Σ-matrix), with $k = x, y, z$,

$$J_k = \int d^3x \; \psi^\dagger(x)\left([x \wedge (-i\nabla)]_k + \frac{1}{2}\Sigma^k\right)\psi(x). \tag{415}$$

Then the quantum generator of rotations gets a hat, and we have included all three directions of rotation, such that $\hat{J}$ is a vector of three operators

$$\hat{J} = \int d^3x \; \hat{\psi}^\dagger(x)\left(x \wedge (-i\nabla) + \frac{1}{2}\Sigma\right)\hat{\psi}(x). \tag{416}$$

Note that the angular momentum are not easy to write in terms of the creation/annihilation operators.

17 Lecture 17: The Quantum Dirac Field, Continued

Recall that fermionic degrees of freedom are described by the quantum field operator

$$\hat{\psi}(x) = \int \frac{d^3p}{(2\pi)^3} \; \frac{1}{\sqrt{2\omega_p}} \sum_{s=1}^{2} \left(\hat{a}_p^s u^s(p) e^{-ip\cdot x} + (\hat{b}_p^s)^\dagger v^s(p) e^{ip\cdot x} \right). \qquad (417)$$

These observable, distribution-valued operators obey the anticommutation relations

$$\{\hat{\psi}_a(x), \hat{\psi}_b^\dagger(y)\} = (2\pi)^3 \delta^{(3)}(x-y) \delta_{ab}. \qquad (418)$$

With this definition, it was postulated that the Hamiltonian is quadratic in the creation-annihilation operators

$$\hat{H} = \int \frac{d^3p}{(2\pi)^3} \; \sum_s \omega_p \left((\hat{a}_p^s)^\dagger \hat{a}_p^s + (\hat{b}_p^s)^\dagger \hat{b}_p^s \right). \qquad (419)$$

This Hamiltonian is a positive operator $\hat{H} \geq 0$ and has a stable vacuum (ground) state $|\Omega\rangle$, such that

$$\hat{a}_p^s |\Omega\rangle = \hat{b}_p^s |\Omega\rangle = 0. \qquad (420)$$

We postulated the generators of rotations, the angular momentum operators, based on the classical analog and the "inverse" Noether's theorem, to be

$$\hat{J} = \int d^3x \; \hat{\psi}^\dagger(x) \left(x \wedge (-i\nabla) + \frac{1}{2}\Sigma \right) \hat{\psi}(x) \qquad (421)$$

Where we recall that each of the quantities $\hat{J}$, x, ∇, and Σ consist of an operator for each spatial dimension x, y, and z.

The next part of our job is to (**Exercise**) check that all of these operators, $\hat{J}$, $\hat{p}$, and boost, the generators of the Poincaré group, obey the correct commutation relations and are Lorentz invariant. In other words, they close to form the Lie algebra of the Poincaré group. This is considerably difficult since these operators are unwieldy in terms of the creation/annihilation operators $\hat{a}$ and $\hat{b}$. See *Weinberg Volume 1* for calculations of the Lie brackets of the Dirac field.

We instead check that the fundamnetal excitations of the DIrac field have spin-$\frac{1}{2}$, which is a weaker assertion than the commutation and Lorentz invariance of the generators, but of interest, nonetheless.

Spin of a Dirac particle at rest

Consider the angular momentum operator acting on one of its eigenstates

$$\hat{J}(\hat{a}_{p=0}^s)^\dagger \ket{\Omega} = [\hat{J}, (\hat{a}_0^s)^\dagger] \ket{\Omega} \tag{422}$$

$$= \int d^3x \left[\hat{\psi}^\dagger(x) \left(x \wedge (-i\nabla) + \frac{1}{2}\Sigma \right) \hat{\psi}(x), (\hat{a}_0^s)^\dagger \right] \ket{\Omega} \tag{423}$$

$$= \int d^3x \, \hat{\psi}^\dagger(x) \left(x \wedge (-i\nabla) + \frac{1}{2}\Sigma \right) \{\hat{\psi}(x), (\hat{a}_0^s)^\dagger\} \ket{\Omega} \tag{424}$$

$$= \int d^3x \, \hat{\psi}^\dagger(x) \left(x \wedge (-i\nabla) + \frac{1}{2}\Sigma \right) \frac{1}{\sqrt{2m}} u^s(p=0) \ket{\Omega} \tag{425}$$

$$= \int d^3x \, \hat{\psi}^\dagger(x) \frac{1}{\sqrt{2m}} \frac{1}{2}\Sigma \, u^s(p=0) \ket{\Omega} \tag{426}$$

Where line 107-108 uses the commutation identity

$$[AB, C] = ABC - CAB = A\{B, C\} \iff \{A, C\} = 0 \tag{427}$$

Line 108-109 uses the anticommutation identity (**Exercise**)

$$\{\hat{\psi}(x), (\hat{a}_{p=0}^s)^\dagger\} = \frac{1}{\sqrt{2\omega_{p=0}}} u^s(p=0) = \frac{1}{\sqrt{2m}} u^s(p=0) \tag{428}$$

And Line 109-110 results since the spatial derivative ∇ acts on $u^s(p=0)$ (4×1 spinor) which is only the dependent on the time component.

Continuing the calculation with the identity (**Exercise**)

$$\int d^3x \, \hat{\psi}^\dagger(x) = \frac{1}{\sqrt{2m}} \left((\hat{a}_0^s)^\dagger (u^s(0))^\dagger + (\hat{b}_0^s)^\dagger (v^s(0))^\dagger \right) \tag{429}$$

And (**Exercise**)

$$(v^r(0))^\dagger \left(\frac{1}{2}\Sigma \, u^s(0) \right) = 0, \; \forall r, \, s \tag{430}$$

We have

$$\hat{J}(\hat{a}_{p=0}^s)^\dagger \ket{\Omega} = \int d^3x \, \hat{\psi}^\dagger(x) \frac{1}{\sqrt{2m}} \frac{1}{2}\Sigma \, u^s(p=0) \ket{\Omega} \tag{431}$$

$$= \frac{1}{\sqrt{2m}} \left((\hat{a}_0^s)^\dagger (u^s(0))^\dagger + (\hat{b}_0^s)^\dagger (v^s(0))^\dagger \right) \frac{1}{\sqrt{2m}} \frac{1}{2}\Sigma \, u^s(p=0) \ket{\Omega} \tag{432}$$

$$= \sum_{r=1}^{2} \left((u^r(0))^\dagger \Sigma^j \frac{1}{2} \frac{1}{\sqrt{2m}} u^s(0) \right) (\hat{a}_0^r)^\dagger \ket{\Omega} \tag{433}$$

Consider the $j = z$ component and the identity $\bar{u}u = 2m$

$$\hat{J}^z (\hat{a}^s_{p=0})^\dagger \ket{\Omega} = \sum_{r=1}^{2} \left((\xi^r)^\dagger \frac{1}{2} \sigma^z \xi^s \right) (\hat{a}^r_0)^\dagger \ket{\Omega} \tag{434}$$

$$= (-1)^{s-1} \frac{1}{2} (\hat{a}^r_0)^\dagger \ket{\Omega} \tag{435}$$

Thus, the eigenvalue of the operator $\hat{J}^z$ on its eigenstate $(\hat{a}^s_{p=0})^\dagger \ket{\Omega}$ is equal to $(-1)^{s-1} \frac{1}{2}$.

Solving the Dirac field

Now we direct our attention to solving the Dirac field via propagators and Wick's theorem. This results is the vacuuum expectation values of products of four-dimensional fermionic field operators, denoted by spinor subscript $a, b = 1, 2, 3, 4$ and boldfaced spacetime coordinate $\hat{\psi}_a(\mathbf{x})$, $\mathbf{x} = (t, x) = (t, x, y, z)$, in terms of Green's functions and Feynman propagators. Note that spatial coordinate and momentum vectors are still denoted by non-boldfaced letters: $x = (x, y, z)$, $p = (p_x, p_y, p_z)$.

$$\bra{\Omega} \hat{\psi}_a(\mathbf{x}) \hat{\bar{\psi}}_b(\mathbf{y}) \ket{\Omega} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} \sum_{s=1}^{2} u^s_a(p) \bar{u}^s_b(p) e^{-ip\cdot(x-y)} \tag{436}$$

$$= (i\slashed{\partial}_{\mathbf{x}} + m)_{ab} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} e^{-ip\cdot(x-y)} \tag{437}$$

$$\tag{438}$$

Similarly (**Exercise**),

$$\bra{\Omega} \hat{\bar{\psi}}_b(\mathbf{y}) \hat{\psi}_a(\mathbf{x}) \ket{\Omega} = (-i\slashed{\partial}_{\mathbf{x}} + m)_{ab} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} e^{-ip\cdot(y-x)}. \tag{439}$$

Spinor field solution to Schroedinger's equation

In order to make sense of these vacuum expectation values, consider the equal-time field operator

$$\hat{\psi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \sum_{s=1}^{2} \left(\hat{a}^s_p u^s(p) e^{-ip\cdot x} + (\hat{b}^s_p)^\dagger v^s(p) e^{ip\cdot x} \right). \tag{440}$$

Using the following commutation relations

$$[\hat{H}, \hat{a}_p^s] = \omega_p \hat{a}_p^s \tag{441}$$

$$[\hat{H}, (\hat{b}_p^s)^\dagger] = -\omega_p (\hat{b}_p^s)^\dagger \tag{442}$$

Add time dependency to the field operator, such that

$$\hat{\psi}(\mathbf{x}) = \hat{\psi}(t, x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \sum_{s=1}^{2} \left(\hat{a}_p^s u^s(p) e^{-i\mathbf{p}\cdot\mathbf{x}} + (\hat{b}_p^s)^\dagger v^s(p) e^{i\mathbf{p}\cdot\mathbf{x}} \right). \tag{443}$$

Combinations of advanced and retarded Green's functions make up the Feynman propagator. For example, construct the retarded Green's function

$$S_{ab}^{ret}(\mathbf{x} - \mathbf{y}) = \theta(x^0 - y^0) \langle \Omega | \{ \hat{\psi}_a(\mathbf{x}), \hat{\bar{\psi}}_b(\mathbf{y}) \} | \Omega \rangle \tag{444}$$

$$= (i\partial\!\!\!/_{\mathbf{x}} + m) D^{ret}(\mathbf{x} - \mathbf{y}) \tag{445}$$

Where $\partial\!\!\!/\partial\!\!\!/ = \Box$ is used to relate back to the bosonic propagator D^{ret} (**Exercise**).

Another way to solve for the fermionic Feynman propagator is to use the Fourier transform (**Exercise**)

$$(i\partial\!\!\!/_{\mathbf{x}} - m) S^{ret}(\mathbf{x} - \mathbf{y}) = i\delta^{(4)}(\mathbf{x} - \mathbf{y}) \cdot \mathbb{I}_{4\times4} \tag{446}$$

$$\downarrow \tag{447}$$

$$\int \frac{d^4p}{(2\pi)^4} (\not{p} - m) e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \tilde{S}^{ret}(\mathbf{p}) = \int \frac{d^4p}{(2\pi)^4} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}. \tag{448}$$

Therefore, in Fourier space, the retarded Green's function is

$$\tilde{S}^{ret}(\mathbf{p}) = \frac{i}{\not{p} - m} = \frac{i(\not{p} + m)}{\not{p}^2 - m^2} = \frac{i(\not{p} + m)}{p^2 - m^2}. \tag{449}$$

Making combinations of $\tilde{S}^{ret}(\mathbf{p})$ and $\tilde{S}^{adv}(\mathbf{p})$, define the Feynman propagator

$$S_F(\mathbf{x} - \mathbf{y}) = \langle \Omega | \mathcal{T}[\hat{\psi}(\mathbf{x})\hat{\bar{\psi}}(\mathbf{y})] | \Omega \rangle \tag{450}$$

$$= \lim_{\epsilon \to 0} \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon} e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \tag{451}$$

18 Lecture 18: Quantum Field Theory for Interacting Fermions and Bosons

The building blocks of a quantum field theory are the observables, n-point Green's functions in the Heisenberg picture

$$G_\alpha^{(n)}(\mathbf{x}_1,\ldots,\mathbf{x}_n) \equiv \langle\Omega|\,\mathcal{T}[\hat{\Phi}_{\alpha_1}(\mathbf{x}_1)\ldots\hat{\Phi}_{\alpha_n}(\mathbf{x}_n)]\,|\Omega\rangle \tag{452}$$

Where $|\Omega\rangle$ is the full interacting vacuum state and α is a vector denoting all relevant quantum fields.

For example, in a theory of scalar bosons and fermions the field operators may be labeled as

$$\hat{\Phi}_0(\mathbf{x}) = \hat{\phi}(\mathbf{x}) \text{ for a scalar boson}$$
$$\hat{\Phi}_\alpha(\mathbf{x}) = \hat{\psi}_\alpha(\mathbf{x}) \text{ for a Dirac spinor, } \alpha = 1,2,3,4$$

We make predictions perturbatively by the general form of the Hamiltonian $\hat{H} = \hat{H}_{free} + \hat{H}_{int}$, where $\hat{H}_{free}$ just contains independent copies of the bosonic and fermionic fields, and $\hat{H}_{int}$ contains cross-terms.

For example, we may have the following free Hamiltonian combining the Klein-Gordon and Dirac fields.

$$\hat{H}_{free} = \hat{H}_{KG} + \hat{H}_{Dirac} \tag{453}$$
$$= \left(\frac{1}{2}\int d^3x\,\hat{\pi}^2(x) + m^2\hat{\phi}^2(x) + (\nabla\hat{\phi}(x))^2\right) + \left(\int d^3x\,\hat{\psi}^\dagger(-i\gamma^0\underline{\gamma}\cdot\underline{\nabla} + m\gamma^0)\hat{\psi}\right). \tag{454}$$

And the interacting Hamiltonian with bosons interacting via the ϕ^4 interaction and bosons and fermions interacting via the second term (Yukawa theory) containing the simplest Lorentz-invariant Dirac spinor quantity $\hat{\bar{\psi}}\hat{\psi}$ (Lorentz scalar from the Lagrangian density). Note that if ϕ^4 is the only interaction, everything decouples.

$$\hat{H}_{int} = \int d^3x\,\frac{\lambda}{4!}\hat{\phi}^4(x) + g\hat{\phi}(x)\hat{\bar{\psi}}(x)\hat{\psi}(x) \tag{455}$$

Recall that the equation used to solve for $G^{(n)}$ is

$$G_\alpha^{(n)}(\mathbf{x}_1,\ldots,\mathbf{x}_n) = \frac{\langle 0|\,\mathcal{T}[\hat{\Phi}_{\alpha_1}(\mathbf{x}_1)\ldots\hat{\Phi}_{\alpha_n}(\mathbf{x}_n)\mathcal{S}]\,|0\rangle}{\langle 0|\,\mathcal{S}\,|0\rangle} \tag{456}$$

Which uses the scattering matrix $\mathcal{S} = \lim_{t\to\infty} U(t,-t)$ (see Lecture 8), which can be expanded perturbatively via the Dyson series

$$U(t, -t) = \mathbb{I} - i \int_{-t}^{t} dt' \hat{H}_{int}(t') + (-i)^2 \int_{-t}^{t} dt' \int_{-t'}^{t'} dt'' \hat{H}_{int}(t') \hat{H}_{int}(t'') + \ldots . \tag{457}$$

After substituting the Dyson series for the scattering matrix, the correlation function is calculated order-by-order, which runs into infinities. We therefore introduced the cutoff for the interacting Hamiltonian up to some scale Λ

$$\hat{H}_{int} \to \hat{H}_{int}(\Lambda). \tag{458}$$

After all, terms in the series for $G^{(n)}$ are time-ordered products of the field operators in the interaction picture $\langle 0| \mathcal{T}[\hat{\Phi}(z_1) \ldots \hat{\Phi}(z_m)] |0\rangle$ and integrals over $z_1, \ldots, z_m$, where Wick's theorem is used to evaluate the time-ordered product. We now generalize Wick's theorem to fermionic (spinor) field operators.

Wick's Theorem for Fermions

Define the time-ordering symbol for fermionic field operators with spinor labels $\alpha, \beta = 1, 2, 3, 4$, and note that the product of the two field operators is actually 16 operators in total.

$$\mathcal{T}[\hat{\psi}_\alpha(\mathbf{x})\hat{\bar{\psi}}_\beta(\mathbf{y})] = \begin{cases} \hat{\psi}_\alpha(\mathbf{x})\hat{\bar{\psi}}_\beta(\mathbf{y}), & x^0 \geq y^0 \\ -\hat{\bar{\psi}}_\beta(\mathbf{y})\hat{\psi}_\alpha(\mathbf{x}), & x^0 < y^0 \end{cases} \tag{459}$$

Recall the Feynman propagator calculated from the time-ordered products contains 16 numbers in total. For two fermionic field operators we can define the time-ordering as

$$S_F^{\alpha\beta} = \int \frac{d^4 p}{(2\pi)^4} \frac{i(\not{p} + m \cdot \mathbb{I})_{\alpha\beta}}{p^2 - m^2 + i\epsilon} e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \equiv \langle 0| \mathcal{T}[\hat{\psi}_\alpha(\mathbf{x})\hat{\bar{\psi}}_\beta(\mathbf{y})] |0\rangle \equiv \overset{\frown}{\hat{\psi}(\mathbf{x})\hat{\bar{\psi}}(\mathbf{y})}. \tag{460}$$

Now we generalize $\mathcal{T}[]$ to many fermionic field operators

$$\mathcal{T}[\hat{\psi}(\mathbf{x}_1) \ldots \hat{\psi}(\mathbf{x}_n)] = (-1)^{sgn(\pi)} \hat{\psi}(\mathbf{x}_{\pi^{-1}(1)}) \ldots \hat{\psi}(\mathbf{x}_{\pi^{-1}(n)}) \tag{461}$$

Where $x^0_{\pi^{-1}(1)} > \cdots > x^0_{\pi^{-1}(n)}$, for $\pi \in S_n$ the symmetric group.

To prove Wick's theorem and produce a workable form of the equation, the base case of Wick's theorem for fermions starts with two field operators, relating the time-ordered product to the normal-ordered product and the Wick contraction

$$\mathcal{T}[\hat{\psi}(\mathbf{x})\hat{\bar{\psi}}(\mathbf{y})] = \mathcal{N}[\hat{\psi}(\mathbf{x})\hat{\bar{\psi}}(\mathbf{y})] + \overset{\frown}{\hat{\psi}(\mathbf{x})\hat{\bar{\psi}}(\mathbf{y})}. \tag{462}$$

Explicitly, the Wick contraction can be written in terms of the anticommutation brackets for positive and negative frequencies (antiparticles) and related to the Feynman propagator (**Exercise**)

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$$\hat{\psi}(\mathbf{x})\hat{\bar{\psi}}(\mathbf{y}) = S_F(\mathbf{x} - \mathbf{y}) = \begin{cases} \{\hat{\psi}^+(\mathbf{x}), \hat{\bar{\psi}}^-(\mathbf{y})\}, \ x^0 \geq y^0 \\ \{\hat{\bar{\psi}}^+(\mathbf{y}), \hat{\psi}^-(\mathbf{x})\}, \ x^0 < y^0. \end{cases} \tag{463}$$

For "non-mixed" field operators, the Wick contraction is zero

$$\hat{\psi}(\mathbf{x})\hat{\psi}(\mathbf{y}) = \hat{\bar{\psi}}(\mathbf{x})\hat{\bar{\psi}}(\mathbf{y}) = 0. \tag{464}$$

Now we extend the normal ordering principle to account for operator interchange, since each interchange of fermionic field operators introduces a minus sign to the product. For example,

$$\mathcal{N}[\hat{\psi}_1\hat{\psi}_2\hat{\bar{\psi}}_3\hat{\bar{\psi}}_4] = (-1)\hat{\psi}_1\hat{\bar{\psi}}_3\mathcal{N}[\hat{\psi}_2\hat{\bar{\psi}}_4] = (-1)S_F(\mathbf{x}_1 - \mathbf{x}_3)\mathcal{N}[\hat{\psi}_2\hat{\bar{\psi}}_4]. \tag{465}$$

WIth this, we extrapolate to the full Wick's theorem, which will be the same as the bosonic case except for minus signs introduced from moving contracted operators next to each other, for fermionic field operators that allows us to calculate time-ordered products of n field operators in terms of the normal-ordered sum of all possible Wick contractions, and, in turn, in terms of Feynman propagators

$$\mathcal{T}[\hat{\psi}_1\hat{\bar{\psi}}_2\dots] = \mathcal{N}[\hat{\psi}_1\hat{\bar{\psi}}_2\dots + \text{"all possible contractions"}]. \tag{466}$$

Schematic of Perturbative QFT Calculation

Begin with the observables and relate them to field operators in the interaction picture

$$G_\alpha^{(n)}(\mathbf{x}_1, \dots, \mathbf{x}_n) = \frac{\langle 0|\, \mathcal{T}[\hat{\Phi}_{\alpha_1}(\mathbf{x}_1)\dots\hat{\Phi}_{\alpha_n}(\mathbf{x}_n)\mathcal{S}]\,|0\rangle}{\langle 0|\,\mathcal{S}\,|0\rangle} \tag{467}$$

Apply the Dyson expansion to the scattering matrix, and then apply Wick's theorem

$$G_\alpha^{(n)}(\mathbf{x}_1, \dots, \mathbf{x}_n) = \mathbb{I} + \int dz \ \hat{\Phi}(z)\hat{\Phi}(z) + \int\int dz \ \hat{\Phi}(z)\hat{\Phi}(z)\hat{\Phi}(z)\hat{\Phi}(z) + \dots \tag{468}$$

$$= \tag{469}$$

The end result if the sum of very many integrals of Wick contractions, and Feynman rules, dependent on the form of $\hat{H}_{int}$, are used to exploit patterns and cancellations in the terms.

Example: Yukawa Theory

Consider the Hamiltonian with interaction based on the fermion density $\hat{\bar{\psi}}\hat{\psi}$ and a single boson field $\hat{\phi}$, make the boson field sensitive to the density of the fermion field

$$\hat{H} = \hat{H}_{KG} + \hat{H}_{Dirac} + \int d^3x \; g \, \hat{\bar{\psi}}(x)\hat{\psi}(x)\hat{\phi}(x). \tag{470}$$

The Feynman rules in momentum space for this quantum field theory are, with dashed lines for bosons and solid lines for fermios

1. Propagators

$$\hat{\phi}(x)\hat{\phi}(y) = \frac{i}{p^2 - m_\phi^2 + i\epsilon} =$$

boson

p

$$\hat{\psi}(x)\hat{\bar{\psi}}(y) = \frac{i(\not{p}+m)}{p^2 - m^2 + i\epsilon} =$$

fermion

p
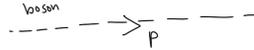
2. Vertices

$$-ig =$$

3. External Legs

$$\hat{\phi}|q\rangle = 1 =$$

boson

q

$$\langle q|\hat{\phi} = 1 =$$

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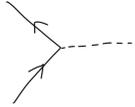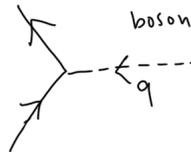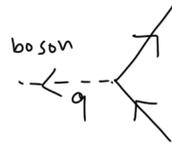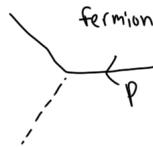
$$\hat{\psi}|p,s\rangle = u^s(p) =$$

$$\langle p,s|\hat{\bar{\psi}} = \bar{u}^s(p) =$$
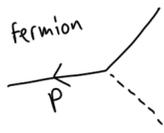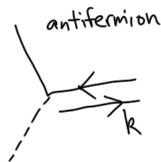
$$\hat{\bar{\psi}}|k,s\rangle = \bar{v}^s(k) =$$

$$\langle k,s|\hat{\psi} = v^s(k) =$$
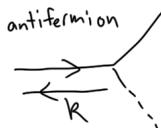
4. Conserve momentum at each vertex

5. Integrate over each loop momentum

6. Calculate the sign of the diagram

7. Divide by the symmetry factor

Then the n-point Green's function is the sum of all connected and amputated Feynman diagrams with n external legs subject to rules 1 through 7 above.
For example, a fermion scattering process looks like

integrate over
x and y, fixed
by conservation

crossing legs
introduces (-1)
(step 6)