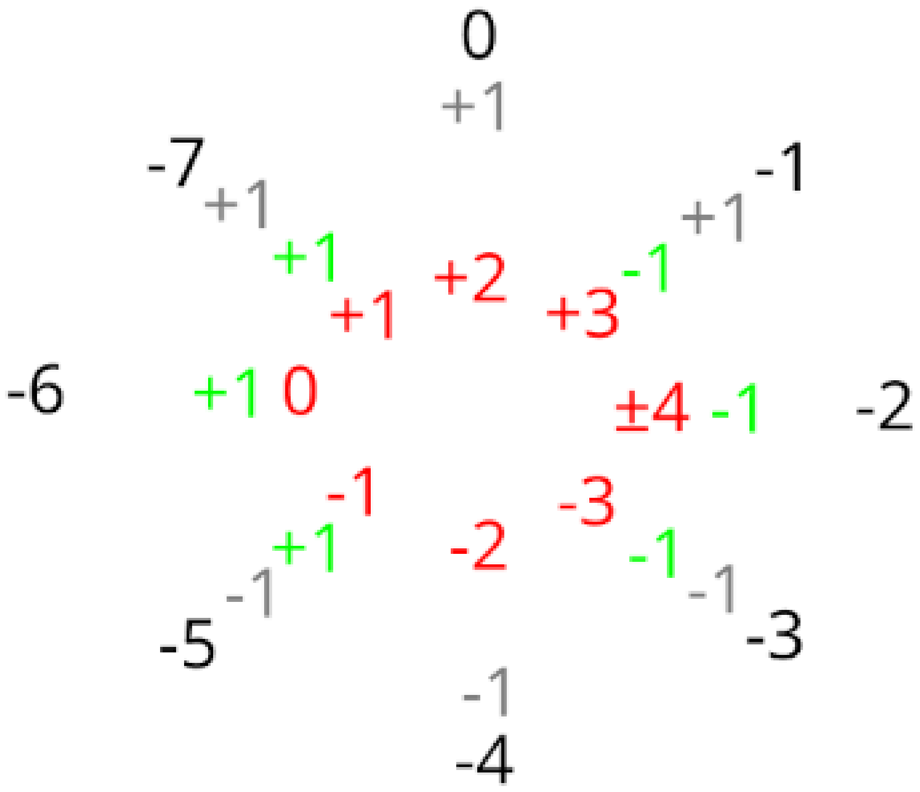


Bott Periodicity for Clifford Algebra Maniacs

Andrius Kulikauskas Math4Wisdom.com 2024.11.11

Clifford algebra

mutually anticommuting
linear complex structures
$J_1, J_2, J_3 \ldots$
imposing commutativity

time reversal T

$T^2 = \pm 1$

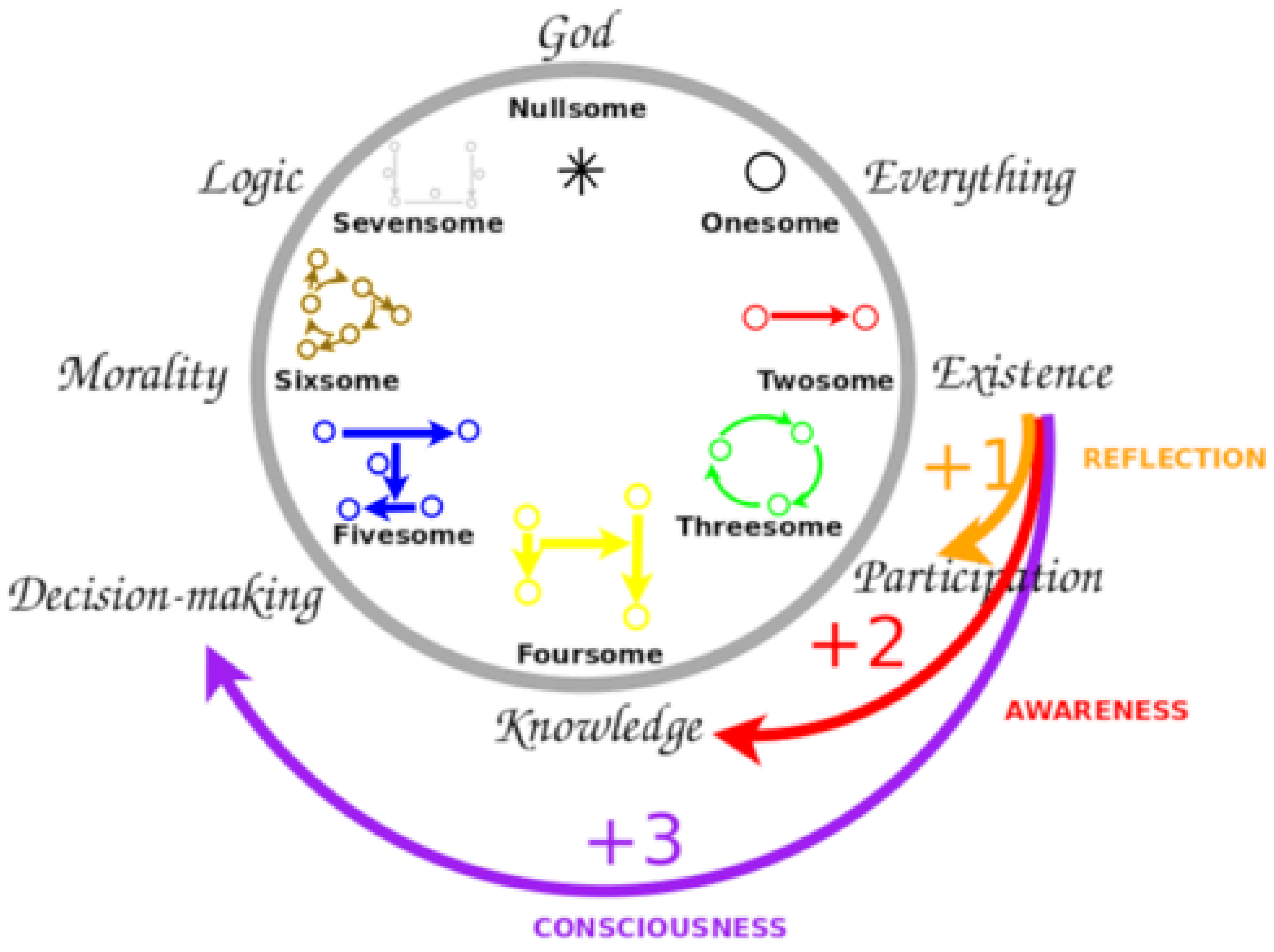
charge conjugation C

$C^2 = \pm 1$

Clifford algebra

mutually anticommuting
implicitly and explicitly complex operators
C, iC, iCT
imposing structure: real, complex, quaternionic

Cognitive frameworks: **Divisions of Everything**



7+3=10 8+2=10 3×3+1=10

0

+1

-7 +1 -1 +1

+1 ℂ+ℂe₋ -1

ℝ+ℝe₋ ℍ+ℍe₊

-6 +1 ℝ ℍ -1 -2

ℝ+ℝe₊ ℍ+ℍe₋

+1 ℂ+ℂe₊ -1

-1 -1

-5 -3

-1

-4

0

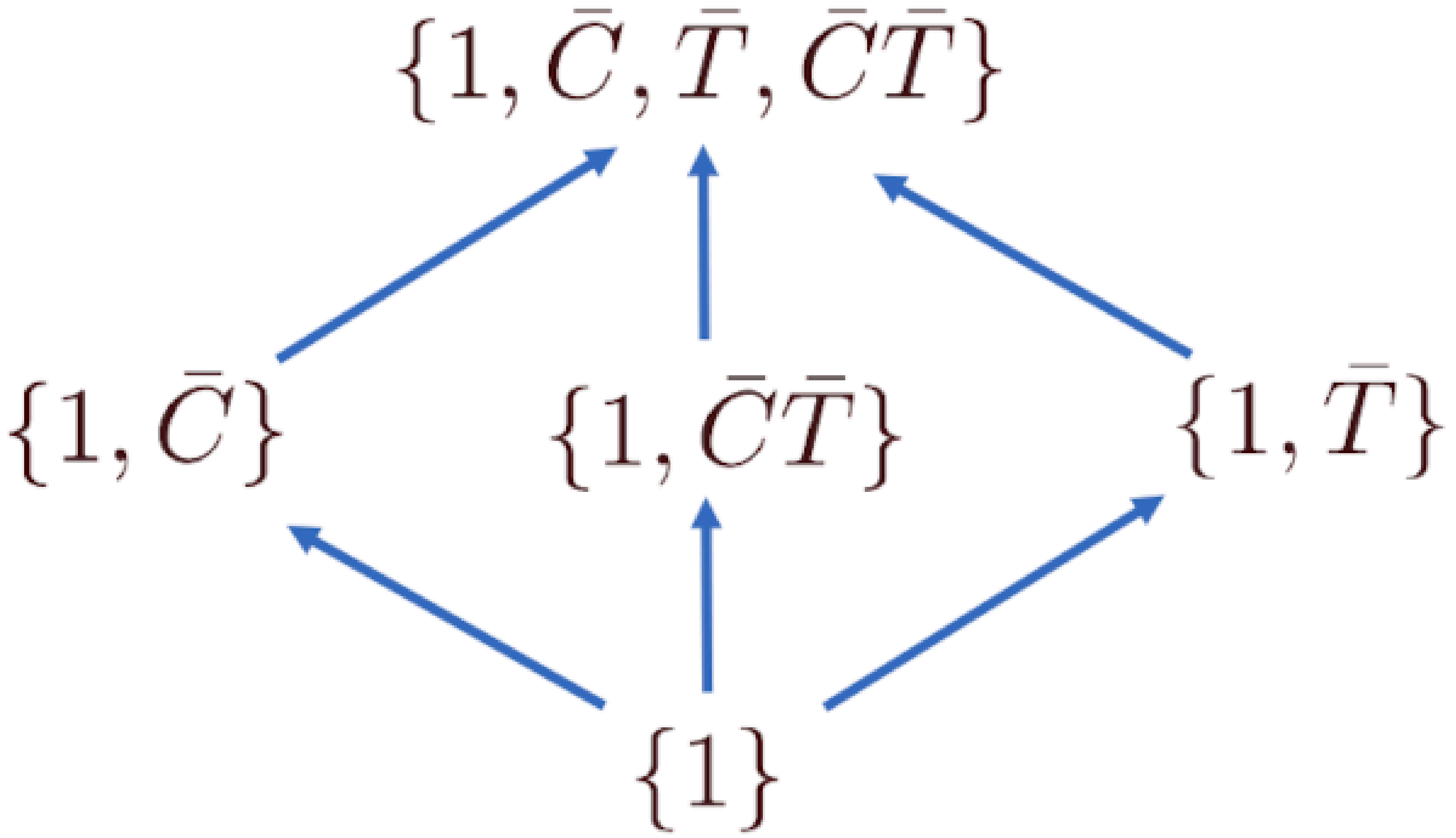
{ }

ℂ

ℂ+ℂe

1 1

1


$\{1, \bar{C}, \bar{T}, \bar{C}\bar{T}\}$
$\{1, \bar{C}\}$
$\{1, \bar{C}\bar{T}\}$
$\{1, \bar{T}\}$
$\{1\}$

$3 \times 3 + 1 = 10$

+1

time reversal T

$T^2 = \pm 1$

+1 +1

-1 +1

{ }

charge conjugation C

$C^2 = \pm 1$

+1

-1

-1 +1

-1 -1

11

-1

8+2=10

0

0

-7

-1

-6

-2

-5

-3

-4

1

Clifford algebra generators

R				1			
C			1		e_1		
H		1		e_1, e_2		$e_1 e_2$	
H+H	1		e_1, e_2, e_3		$e_1 e_2, e_1 e_3, e_2 e_3$		$e_1 e_2 e_3$

Multiplying Clifford algebra generators

$$-1 = e_1^2 = e_2^2 = e_3^2 \ldots$$

$$e_1 e_2 = -e_2 e_1$$

$$(e_1 e_2)^2 = e_1 e_2 e_1 e_2 = -e_1 e_1 e_2 e_2 = -1$$

$$(e_1 e_2 e_3)^2 = e_1 e_2 e_3 e_1 e_2 e_3 = e_1 e_1 e_2 e_3 e_2 e_3 = -e_1 e_1 = +1$$

$7+3=10$

$\mathbb{C}+\mathbb{C}e_-$

$\mathbb{C}$

$\mathbb{R}+\mathbb{R}e_-$ $\mathbb{H}+\mathbb{H}e_+$

$\mathbb{R}$

$\mathbb{H}$

$\mathbb{R}+\mathbb{R}e_+$ $\mathbb{H}+\mathbb{H}e_-$

$\mathbb{C}+\mathbb{C}e_+$

$\mathbb{C}+\mathbb{C}e$

$$
\begin{aligned}
\mathrm{Cliff}_0 &\simeq \mathbb{R} \\
\mathrm{Cliff}_1 &\simeq \mathbb{R} + \mathbb{R}e, \quad e^2 = -1 \\
\mathrm{Cliff}_2 &\simeq \mathbb{C} + \mathbb{C}e, \quad e^2 = -1, ei = -ie \\
\mathrm{Cliff}_3 &\simeq \mathbb{H} + \mathbb{H}e, \quad e^2 = 1, ei = ie, ej = je, ek = ke \\
\mathrm{Cliff}_4 &\simeq \mathbb{H} \\
\mathrm{Cliff}_5 &\simeq \mathbb{H} + \mathbb{H}e, \quad e^2 = -1, ei = ie, ej = je, ek = ke \\
\mathrm{Cliff}_6 &\simeq \mathbb{C} + \mathbb{C}e, \quad e^2 = 1, ei = -ie \\
\mathrm{Cliff}_7 &\simeq \mathbb{R} + \mathbb{R}e, \quad e^2 = 1
\end{aligned}
$$

The key observation is that for any $a \in A$, there exists a unique $a' \in A$ such that

$$ae = ea'$$

and that the A-bimodule structure forces $(ab)' = a'b'$. Hence we have an automorphism (fixing the real field)

$$(-)' : A \to A$$

and we can easily enumerate (up to isomorphism) the possibilities for associative division superalgebras over $\mathbb{R}$:

1. $A = \mathbb{R}$. Here we can adjust e so that $e^2 := \langle e, e \rangle$ is either -1 or 1. The corresponding division superalgebras occur at 1 o'clock and 7 o'clock on the super Brauer clock.
2. $A = \mathbb{C}$. There are two $\mathbb{R}$-automorphisms $\mathbb{C} \to \mathbb{C}$. In the case where the automorphism is conjugation, condition $(\star)$ for super associativity gives $\langle e, e \rangle e = e \langle e, e \rangle$ so that $\langle e, e \rangle$ must be *real*. Again e can be adjusted so that $\langle e, e \rangle$ equals -1 or 1. These possibilities occur at 2 o'clock and 6 o'clock on the super Brauer clock.

   For the identity automorphism, we can adjust e so that $\langle e, e \rangle$ is 1. This gives the super algebra $\mathbb{C}[e]/\langle e^2 - 1 \rangle$ (where e commutes with elements in $\mathbb{C}$). This does not occur on the super Brauer clock over $\mathbb{R}$. However, it does generate the super Brauer group over $\mathbb{C}$ (which is of order two).

3. $A = \mathbb{H}$. Here $\mathbb{R}$-automorphisms $\mathbb{H} \to \mathbb{H}$ are given by $h \mapsto xhx^{-1}$ for $x \in \mathbb{H}$. In other words

   $$he = exhx^{-1}$$

   whence ex commutes with all elements of $\mathbb{H}$ (i.e. we can assume wlog that the automorphism is the identity). The properties of the pairing guarantee that $h\langle e, e \rangle = \langle e, e \rangle h$ for all $h \in \mathbb{H}$, so $\langle e, e \rangle$ is real and again we can adjust e so that $\langle e, e \rangle$ equals 1 or -1. These cases occur at 3 o'clock and 5 o'clock on the super Brauer clock.

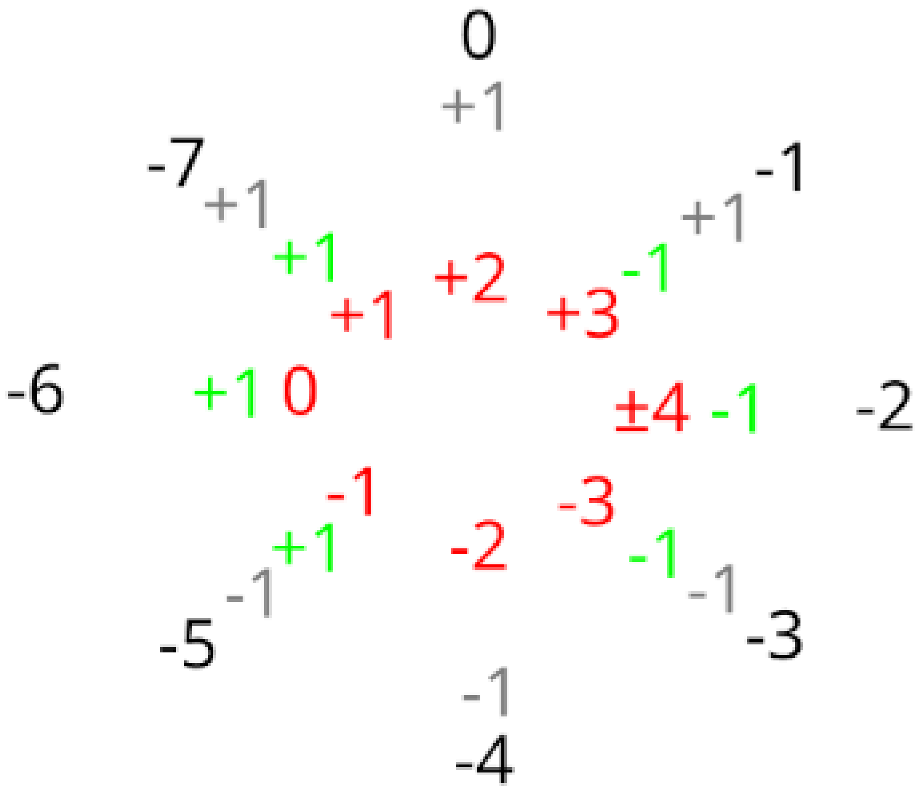
Chomsky hierarchy of automata

Finite automata: $re = er$ implies $e \to er$

Pushdown automata: $ce = e\bar{c}$ implies $e \to cec$

Linear bounded Turing machine: $hex = exh$ implies $hhhhhhex \to exhhhhhh$

Unbounded Turing machine: $e \to$



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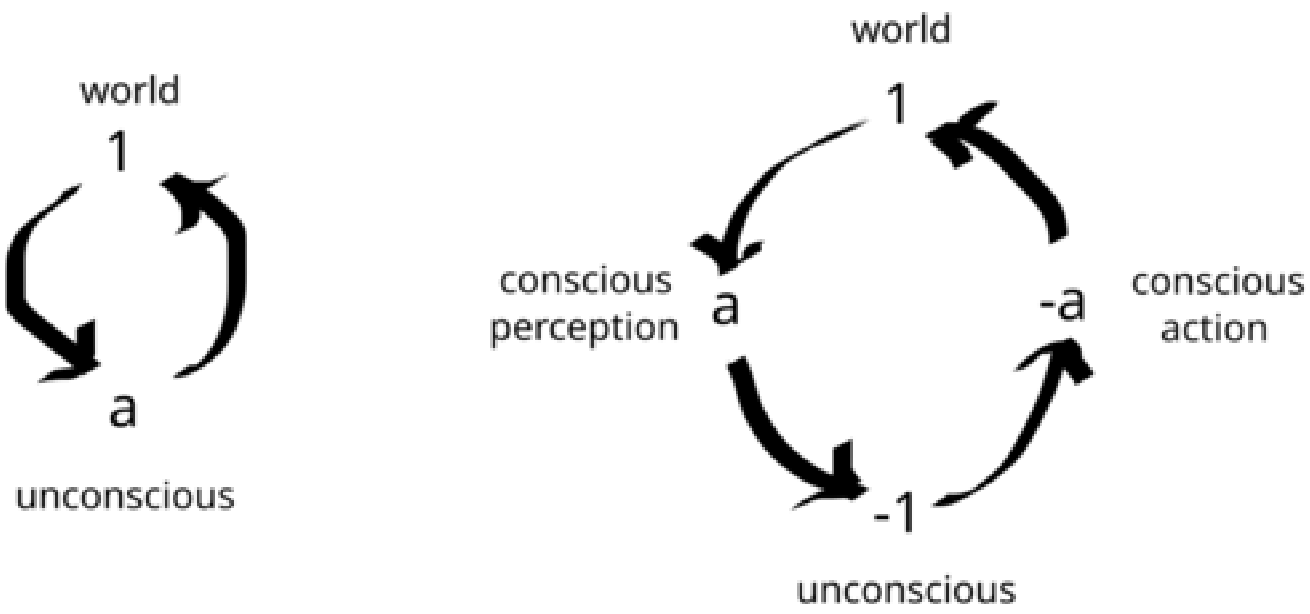
$$a^4 = 1 \quad \text{implies} \quad a^2 = \pm 1$$

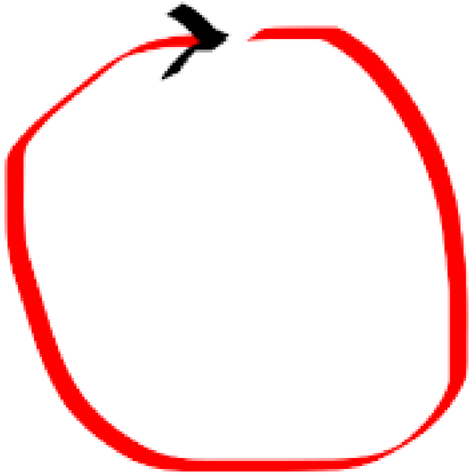


$$1 \to U(1) \xrightarrow{\iota} G \xrightarrow{g} U \to 1$$

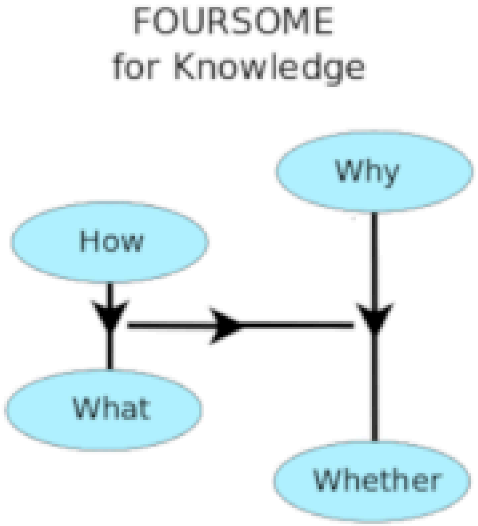
z

a

$$a^2 \in \iota(U(1))$$



recurring activity



structure

$$1 \to U(1) \xrightarrow{\iota} G \xrightarrow{g} U \to 1 \qquad U = \mathbb{Z}_2$$

z

a

$$a^2 \in \iota(U(1))$$

commutative

$aza^{-1} = z$

$az = za$

$$G \cong U(1) \times \mathbb{Z}_2$$

noncommutative

$aza^{-1} = z^{-1}$

$az = z^{-1}a$

$a^4 = 1$

$a^2 = 1$

$$\mathrm{Pin}_+(2)$$

like dihedral

$a^2 = -1$

$$\mathrm{Pin}_-(2)$$

like dicyclic

(ϕ, χ)-representations of CT-groups

Identity: linear, even: $\rho(I) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Time reversal, antilinear, even: $\rho(\bar{T}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Charge conjugation, antilinear, odd: $\rho(\bar{C}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Parity, linear, odd: $\rho(\bar{CT}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Subgroup $U \subset M_{2,2}$	$\tilde{T}^2$	$\tilde{C}^2$	[Clifford]
$\{1\}$			$[\mathbb{C}\ell_0] = [\mathbb{C}]$
$\{1, \bar{S}\}$			$[\mathbb{C}\ell_1]$
$\{1, \bar{T}\}$	$+1$		$[C\ell_0] = [\mathbb{R}]$
$M_{2,2}$	$+1$	-1	$[C\ell_{-1}]$
$\{1, \bar{C}\}$		-1	$[C\ell_{-2}]$
$M_{2,2}$	-1	-1	$[C\ell_{-3}]$
$\{1, \bar{T}\}$	-1		$[C\ell_4] = [\mathbb{H}]$
$M_{2,2}$	-1	$+1$	$[C\ell_{+3}]$
$\{1, \bar{C}\}$		$+1$	$[C\ell_{+2}]$
$M_{2,2}$	$+1$	$+1$	$[C\ell_{+1}]$

Moore **CT Groups**

Stone, Roy, Chiu
Symmetric spaces

Cartan	TRS	PHS	SLS	Hamiltonian $M = G/H$	Classifying Q
D	0	+1	0	$\mathrm{O}(16r) \times \mathrm{O}(16r)/\mathrm{O}(16r) \simeq \mathrm{O}(16r)$	R_2
DIII	−1	+1	1	$\mathrm{O}(16r)/\mathrm{U}(8r)$	R_3
AII	−1	0	0	$\mathrm{U}(8r)/\mathrm{Sp}(4r)$	R_4
CII	−1	−1	1	$\{\mathrm{Sp}(4r)/\mathrm{Sp}(2r) \times \mathrm{Sp}(2r)\} \times \mathbb{Z}$	R_5
C	0	−1	0	$\mathrm{Sp}(2r) \times \mathrm{Sp}(2r)/\mathrm{Sp}(2r) \simeq \mathrm{Sp}(2r)$	R_6
CI	+1	−1	1	$\mathrm{Sp}(2r)/\mathrm{U}(2r)$	R_7
AI	+1	0	0	$\mathrm{U}(2r)/\mathrm{O}(2r)$	R_0
BDI	+1	+1	1	$\{\mathrm{O}(2r)/\mathrm{O}(r) \times \mathrm{O}(r)\} \times \mathbb{Z}$	R_1
D	0	+1	0	$\mathrm{O}(r) \times \mathrm{O}(r)/\mathrm{O}(r) \simeq \mathrm{O}(r)$	R_2

Clifford Algebra	Ungraded algebra	Graded algebra	Ungraded irreps	Graded irreps
$C\ell_{+4}$	$\mathbb{H}(2)$	$\mathrm{End}(\mathbb{R}^{1\|1})\widehat{\otimes}\mathbb{H}$	$\mathbb{H}^2$	$\tilde{\mu}^{\pm}$
$C\ell_{+3}$	$\mathbb{C}(2)$	$\mathbb{R}[\varepsilon_-]\widehat{\otimes}\mathbb{H}$	$\mathbb{C}^2$	$\tilde{\eta}^3$
$C\ell_{+2}$	$\mathbb{R}(2)$	$\mathbb{C}[\varepsilon_+]$, $z\varepsilon_+ = \varepsilon_+\bar{z}$	$\mathbb{R}^2$	$\tilde{\eta}^2$
$C\ell_{+1}$	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{R}[\varepsilon_+]$	$\mathbb{R}_{\pm}, \rho(e) = \pm 1$	$\tilde{\eta}$
$C\ell_0$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}^{1\|0}, \mathbb{R}^{0\|1}$
$C\ell_{-1}$	$\mathbb{C}$	$\mathbb{R}[\varepsilon_-]$	$\mathbb{C}$	η
$C\ell_{-2}$	$\mathbb{H}$	$\mathbb{C}[\varepsilon_-]$, $z\varepsilon_- = \varepsilon_-\bar{z}$	$\mathbb{H}$	η^2
$C\ell_{-3}$	$\mathbb{H}\oplus\mathbb{H}$	$\mathbb{R}[\varepsilon_+]\widehat{\otimes}\mathbb{H}$	$\mathbb{H}_{\pm}$, $\rho(e_1e_2e_3) = \pm 1$	η^3
$C\ell_{-4}$	$\mathbb{H}(2)$	$\mathrm{End}(\mathbb{R}^{1\|1})\widehat{\otimes}\mathbb{H}$	$\mathbb{H}^2$	$\mu^{\pm}$

Notation

$\varepsilon\pm$ is odd and $\varepsilon^2\pm = \pm 1$

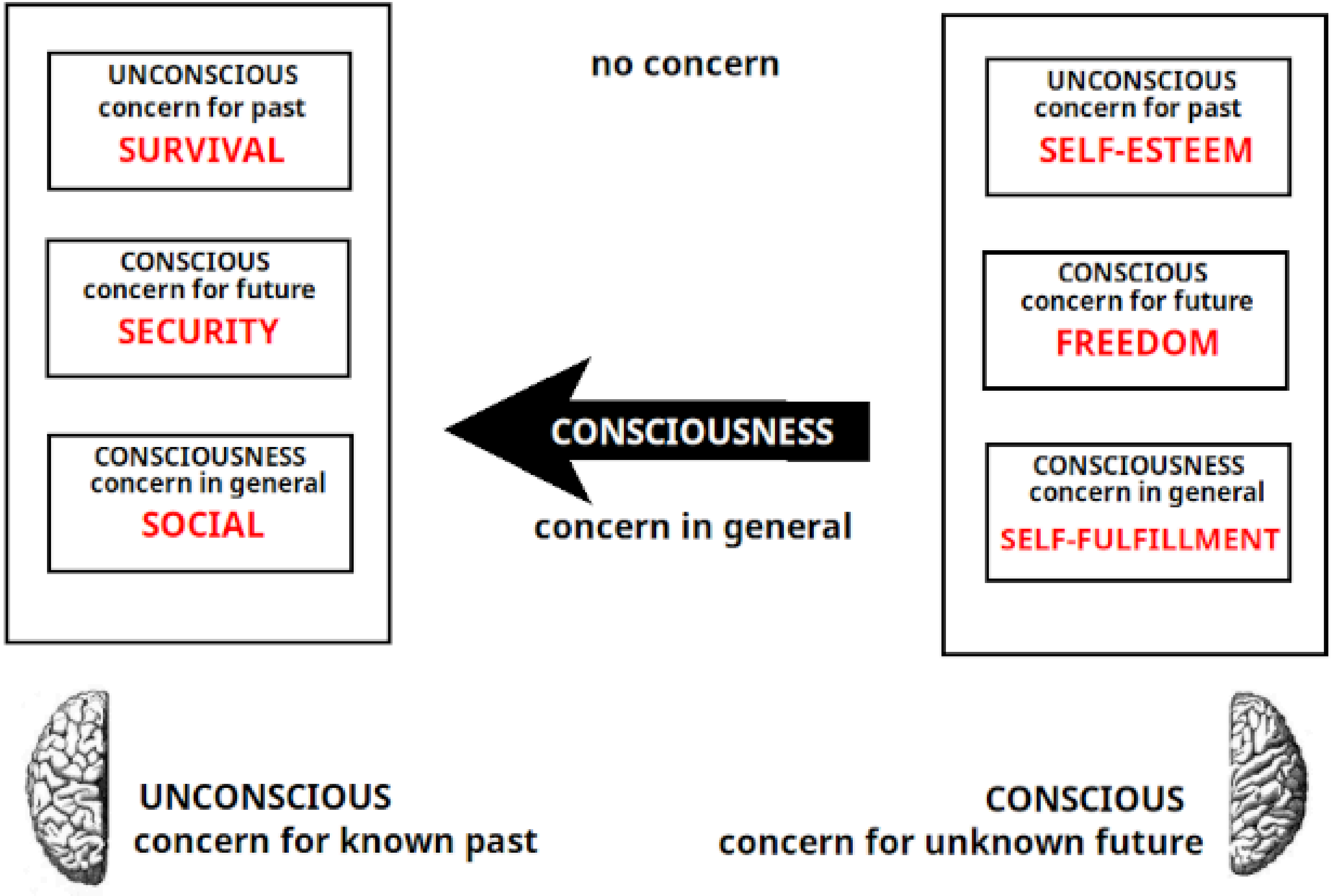
$$\eta = \mathbb{R}^{1|1} \quad \rho(e) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \tilde{\eta} = \mathbb{R}^{1|1} \quad \rho(e) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\iota = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \varphi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \psi = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

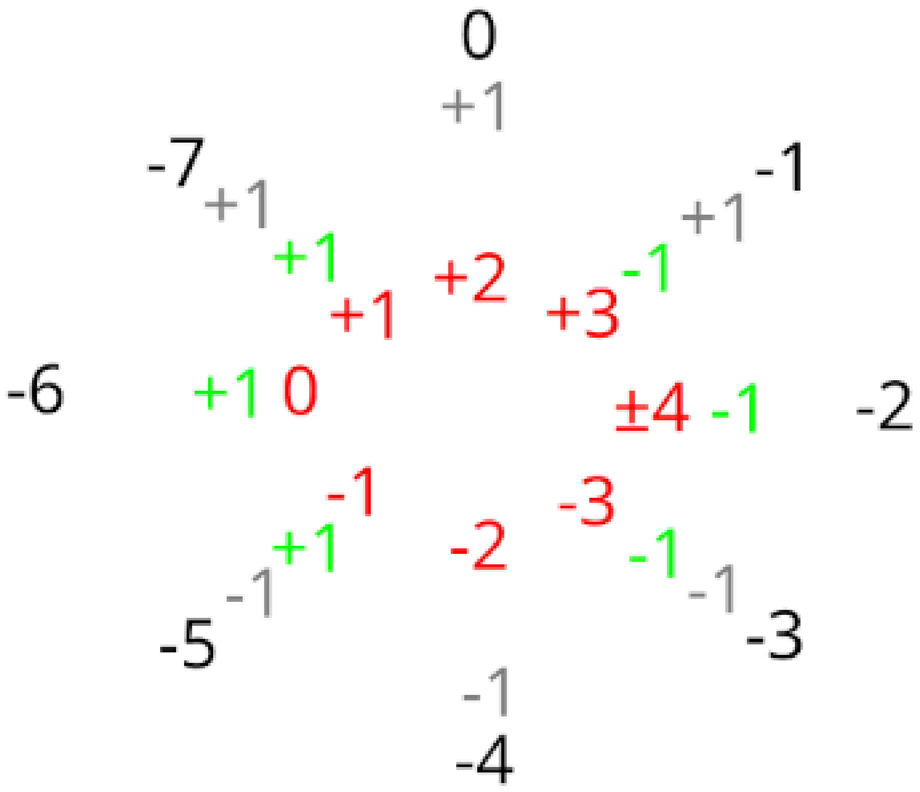
Lie algebra decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ where $[\mathfrak{h}, \mathfrak{h}] \in \mathfrak{h}$, $[\mathfrak{h}, \mathfrak{m}] \in \mathfrak{m}$, $[\mathfrak{m}, \mathfrak{m}] \in \mathfrak{h}$.

$m \in G/H$

				symmetric space G/H	Hamiltonian	T^2	C^2	T	C	Restrict
iC	$C\ell_{+2}$	$\iota \equiv j_1$	$C\ell_0$	$O(16r)\times O(16r)/O(16r)$	$\iota \otimes m$		+1		$\varphi \otimes \mathbb{I}$	
iCT	$C\ell_{+3}$	$J_1 = \mathrm{diag}(\iota)$	$C\ell_{-1}$	$O(16r)/U(8r)$	$\iota \otimes m$	-1	+1	$\varphi \otimes J_1$	φ	
$CT?$	$C\ell_4$	$J_2 = \mathrm{diag}(\psi)$	$C\ell_{-2}$	$U(8r)/Sp(4r)$	$J_1 m$	-1		J_2		
iCT	$C\ell_{-3}$	$J_3^{-1} = I_1 J_1 J_2$	$C\ell_{-3}$	$Sp(4r)/Sp(2r)\times Sp(2r)$	$J_1 m$	-1	-1	J_3	J_2	
iC	$C\ell_{-2}$	$J_4^{-1} = LJ_3$	$C\ell_{-4}$	$Sp(2r)\times Sp(2r)/Sp(2r)$	$J_1 m$		-1		J_2	$J_1 J_2 J_3 = +1$
$C\vert_{T=+1}$	$C\ell_{-1}$	$J_5^{-1} = I_2 J_1 J_4$	$C\ell_{-5}$	$Sp(2r)/U(2r)$	$J_1 m$	+1	-1	$J_2 J_4 J_5$	J_2	
$\vert_{T=+1}$	$C\ell_0$	$J_6^{-1} = I_3 J_2 J_4$	$C\ell_{-6}$	$U(2r)/O(2r)$	$J_1 m$	+1		$J_3 J_4 J_6$ or $J_2 J_4 J_6$		$J_1 J_4 J_5 = +1$
$C\vert_{T=+1}$	$C\ell_{+1}$	$J_7^{-1} = I_4 J_1 J_6$	$C\ell_{-7}$	$O(2r)/O(r)\times O(r)$	$J_1 m$	+1	+1	$J_1 J_6 J_7$	$J_2 J_4 J_6 \equiv \varphi$	
iC	$C\ell_{+2}$	$J_8^{-1} = L_2 J_7$	$C\ell_{-8}$	$O(r)\times O(r)/O(r)$	$J_1 m$		+1		$J_2 J_4 J_6 \equiv \varphi$	$J_2 J_4 J_6,\ J_1 J_6 J_7 = +1$



UNCONSCIOUS
concern for past
SURVIVAL
CONSCIOUS
concern for future
SECURITY
CONSCIOUSNESS
concern in general
SOCIAL
no concern
CONSCIOUSNESS
concern in general
UNCONSCIOUS
concern for past
SELF-ESTEEM
CONSCIOUS
concern for future
FREEDOM
CONSCIOUSNESS
concern in general
SELF-FULFILLMENT
UNCONSCIOUS
concern for known past
CONSCIOUS
concern for unknown future



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