What are Complex Numbers and why are they useful?

(7/24/23)

Informal definition:

The Complex Number System is what results by allowing square roots of negative numbers.
Complex numbers are of the form

$$\underbrace{a}_{\text{Real Part}} + i\underbrace{b}_{\text{Imag Part}}$$

where a and b are real numbers and $i = \sqrt{-1}$ "the imaginary unit"

$$i^2 = -1$$

Ex: $2+3i$ $\quad$ $\underbrace{3}_{3+0i}$ "pure real" $\quad$ $\underbrace{-4i}_{0+(-4)i}$ "pure imaginary"

No real numbers can be the square root of -1!

why? If a is positive then $a^2 > 0$ Ex: $1^2 = 1 > 0$

If a is negative then $a^2 > 0$. Ex: $(-1)^2 = 1 > 0$

Thus, if it exists i is not a "real number". This lead Descarte, in 1637, to call it an "imaginary" number.

Note: the square root of any negative numbers can be written in terms of i.

Ex: $\sqrt{-4} = 2i$ why? $(2i)^2 = 2^2 \cdot i^2 = 4(-1)$
$= -4$

$\sqrt{-\frac{8}{7}} = i\sqrt{\frac{8}{7}} =$

In general: if $a > 0$ then $\sqrt{-a} = i\sqrt{a}$

Why are we interested in the square roots of negative numbers?

Historically, one context in which these arose was in considering solutions to quadratic equations:

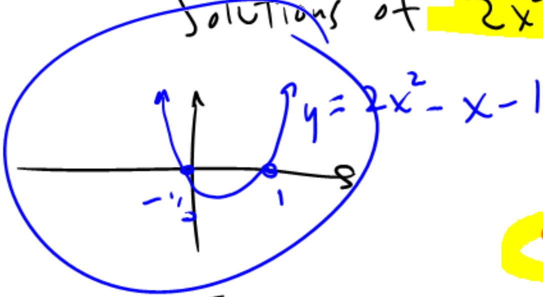
Solutions of $ax^2+bx+c=0$ are

$$\boxed{x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}}$$

Ex:

Solutions of $2x^2-x-1=0$: $x = \dfrac{1 \pm \sqrt{1-(-8)}}{4}$

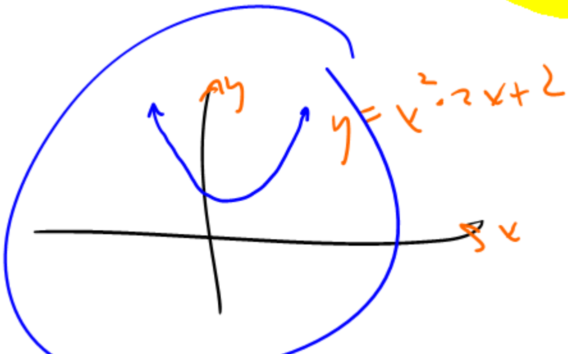
$= -\frac{1}{2}, 1$



check: $2\left(-\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right) - 1 = \frac{1}{2} + \frac{1}{2} - 1 = 0$ ✓

Solutions of x^2-2x+2: $x = \dfrac{2 \pm \sqrt{4-8}}{2}$

$= 1+i, \ 1-i$



Who cares? Do these complex solutions have any significance?

Ex: Suppose we have 200 ft of fencing. Can we use it to surround a rectangular enclosure with area

(1) 2000 ft^2?

(2) 3000 ft^2?

Solution of (1):

$$\text{Area} = 2000$$

$$x(100-x) = 2000$$

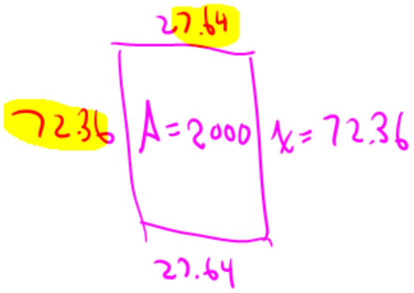
$$x^2 - 100x + 2000 = 0$$

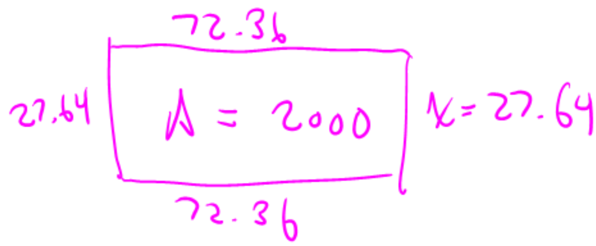
$$x = \frac{100 \pm \sqrt{100^2 - 8000}}{2}$$

$b^2 - 4ac > 0$

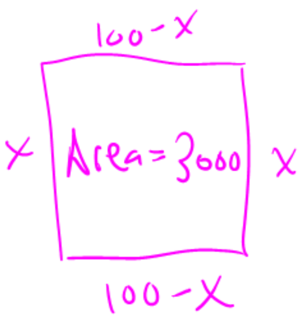
$\approx 72.36,\ 27.64$

yes it can be done. Two solutions





(2)



$$\text{Area} = 3000$$

$$x(100-x) = 3000$$

$$x^2 - 100x + 3000 = 0 \quad (b^2 - 4ac < 0)$$

$$x = \frac{100 \pm \sqrt{100^2 - 12{,}000}}{2} \quad \cancel{= 50 \pm 10i\sqrt{5}}$$

What does this mean?

You can't do it!

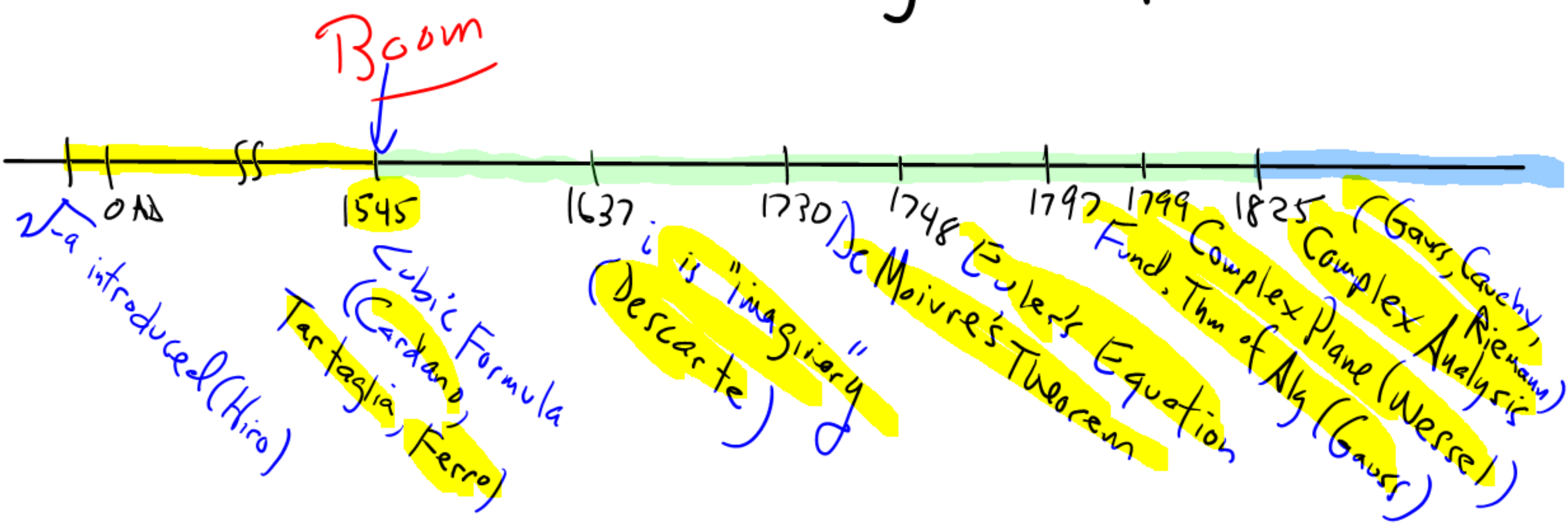
So... can't we just say:

Square roots of negative numbers aren't allowed. If you encounter them, just throw them out and say: NO SOLUTION

? ? ? ? ? ?

For the first 1500 years of i, this was primarily how it was dealt with. Then, in 1545...

Rough timeline of the history of complex numbers:



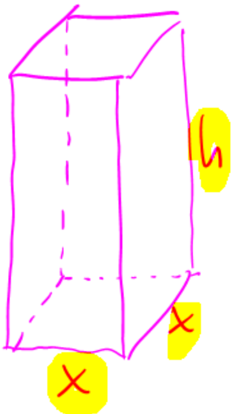
What happened in 1545?

Cardano's Formula for the solutions to cubic equations:

Solutions of $x^3 + qx + p = 0$ are of the form

$$\boxed{x = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}}$$

Ex: Suppose we want to build a rectangular box with a square base, and with total area $= 100 \text{ in}^2$ and volume $= 50 \text{ in}^3$. Can this be done? If so, what are the dimensions of the box?



Area = 100

$$2x^2+4xh = 100$$

$$h = \frac{50-x^2}{2x}$$

Volume = 50

$$x^2h = 50$$

$$x^2\left(\frac{50-x^2}{2x}\right) = 50$$

$$x^3-50x+100=0$$

$p=-50$ $q=100$

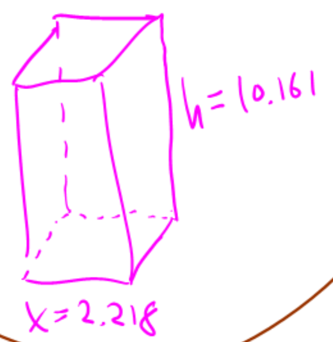
Cardano's Formula:

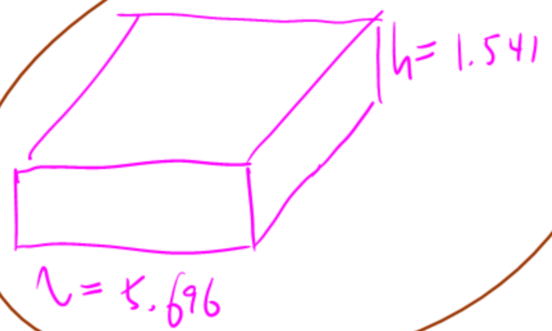
$$x = \sqrt[3]{-\frac{100}{2}+\sqrt{\frac{100^2}{4}+\frac{(-50)^3}{27}}} + \sqrt[3]{-\frac{100}{2}-\sqrt{\frac{100^2}{4}+\frac{(-50)^3}{27}}}$$

$$= \sqrt[3]{50}\left[\sqrt[3]{-1+i\sqrt{\frac{23}{27}}} + \sqrt[3]{-1-i\sqrt{\frac{23}{27}}}\right]$$

$$\approx 5.696,\ 2.218,\ \cancel{-7.91}$$

Two Solutions:





Cardano's Formula often requires use of complex numbers in intermediate computations, even if the solutions are real!

Starting with Cardano's formula, complex numbers could no longer be ignored!

Back to the definition of Complex Numbers...
Is it really ok to add $i = \sqrt{-1}$ into our number system? Could it goof anything up?

Warning: You can't just throw anything you want into a number system!

Ex: Define $k = \frac{1}{0}$. Is it ok to treat k like a number?

If k is a number then:

$$0 \cdot k = \cancel{0} \cdot \frac{1}{\cancel{0}} = 1$$

$$0 \cdot \text{anything} = 0$$

thus $\boxed{0 = 1}$

But if this is true than all numbers are equal!

$$\begin{array}{r} 5 = 5 \\ + \; 0 = 1 \\ \hline 5 = 6 \\ + \; 0 = 1 \\ \hline 5 = 7 \end{array}$$

every number = every number

Upshot: introducing $k = \frac{1}{0}$ into a number system totally goofs it up!
The number system becomes inconsistent.

How do we know this doesn't happen when we introduce i into our number system?

Answer: We approach the Complex Number system from a different direction.

Formal Definition of the Complex Number System:

The Complex Numbers are all ordered pairs (a,b) where a and b are real numbers and:

(1) Addition is defined by
$$(a,b)+(c,d) = (a+c, b+d)$$

(2) Multiplication is defined by
$$(a,b)\cdot(c,d) = (ac-bd, ad+bc)$$

$(a+bi)(c+di) = ac + bd(i^2) + adi + bci = (ac-bd) + (ad+bc)i$

That's it!

Wait! How are these numbers?

To be continued

Aren't these pairs of numbers?

Continuing 9/29/23

What is meant by "a number system"?

Answer: in mathematics a number system is any set of objects that satisfies the following "field" axioms:

Properties of a Field:

Wikipedia: Field (mathematics)

- Associativity of addition and multiplication: $a + (b + c) = (a + b) + c$, and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
- Commutativity of addition and multiplication: $a + b = b + a$, and $a \cdot b = b \cdot a$.
- Additive and multiplicative identity: there exist two different elements 0 and 1 in F such that $a + 0 = a$ and $a \cdot 1 = a$.
- Additive inverses: for every a in F, there exists an element in F, denoted $-a$, called the *additive inverse* of a, such that $a + (-a) = 0$.
- Multiplicative inverses: for every $a \neq 0$ in F, there exists an element in F, denoted by a^{-1} or $1/a$, called the *multiplicative inverse* of a, such that $a \cdot a^{-1} = 1$.
- Distributivity of multiplication over addition: $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.

In the Complex numbers:

additive identity is $(0,0)$

multiplicative identity is $(1,0)$

additive inverse of (a,b) is $(-a,-b)$

multiplicative inverse of (a,b) is $\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$ etc...

Shorthand notation:

$(1,0) \longrightarrow 1$

$(0,1) \longrightarrow i$

$(a,0) \longrightarrow a$

$(0,b) \longrightarrow bi$

$(a,b) \longrightarrow a+bi$

Note: $i^2 = (0,1)\cdot(0,1) = (0-1, 0+0) = (-1,0) = -1$

thus: $i^2 = -1$ by construction!

.... etc...

The Complex Numbers are Consistent:

Another approach

Complex #s are all 2x2 matrices of form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ $a, b \in \mathbb{R}$ is a field!

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \longrightarrow 1$$

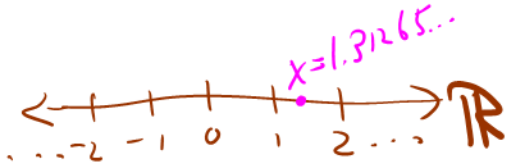
$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \longrightarrow i$$

etc...

$$i^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -1$$

How do we visualize the Complex Numbers?

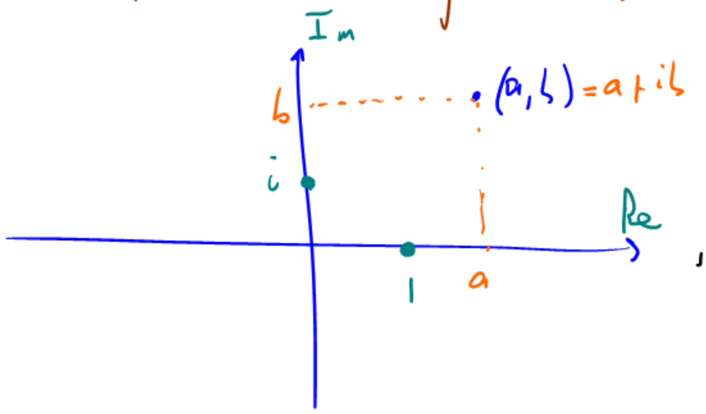
Real Numbers



The number line

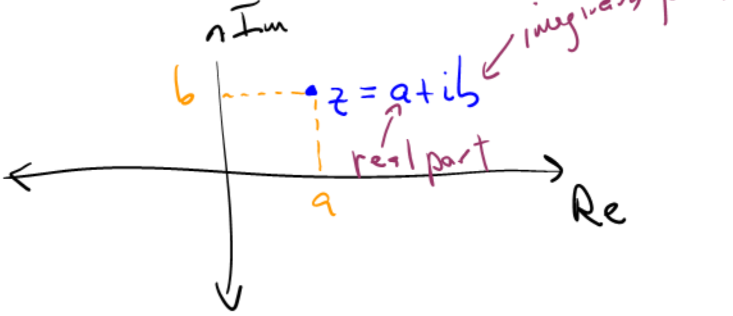
Complex Numbers

all ordered pairs (a, b)

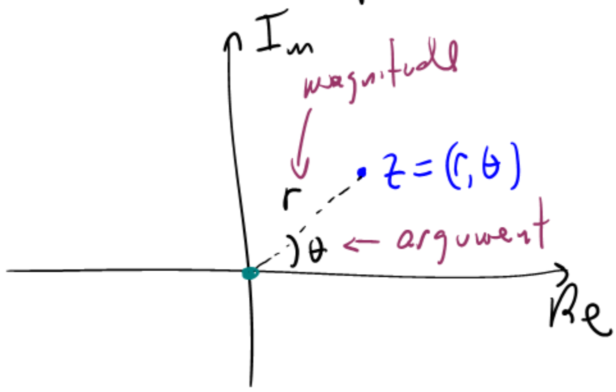


Two ways of representing complex numbers

Standard (rectangular) Representation

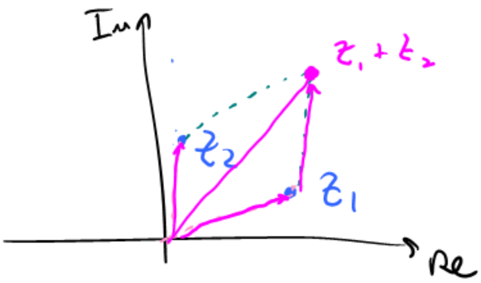


Polar Representation



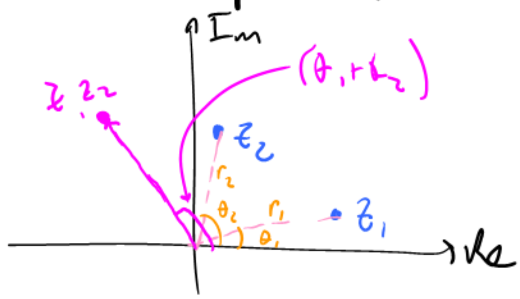
Addition and multiplication of numbers in the Complex Plane

Addition



"vector addition"

Multiplication



add angles
multiply magnitudes.

So that's what complex numbers are.

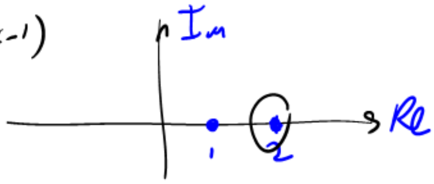
What are they used for?

- Needed for certain computations
  - Cardano's formula, for example
- Fundamental Theorem of Algebra (Gauss 1797):

Every polynomial of degree n has exactly n zeros (counting multiplicity) in the complex plane.

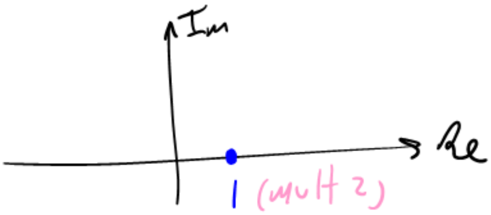
Ex: $x^2 - 3x + 2$ has zeros at $x = 1, 2$

$= (x-2)(x-1)$

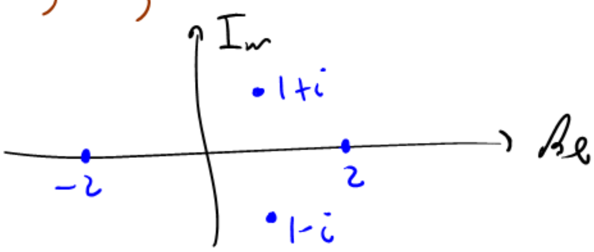


Ex: $x^2 - 2x + 1$ has a zero of multiplicity 2 at $x = 1$

$= (x-1)^2$



Ex: $x^4 - 2x^3 - 2x^2 + 8x - 8$ has zeros at $2, -2, 1+i$ and $1-i$



Authentic Applications of complex numbers (of interest to me):

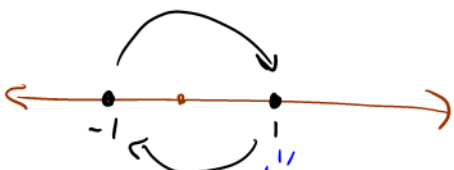
Complex numbers arise naturally in models of Periodic Phenomona, including

- Vibrations
- Waves
- Orbits
- Seasons
- any cyclical behavior

why?

Real Numbers

$\mathbb{R}$



No "natural" cycles other than

x	-1
x^2	1
x^3	-1
x^4	1
x^5	-1

etc

2-cycle

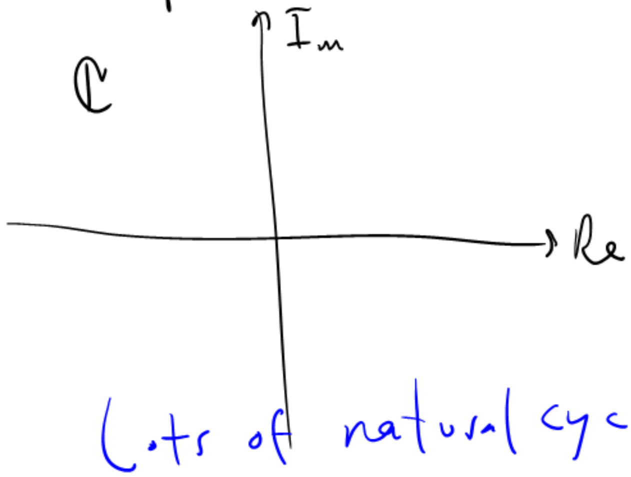
Continuous Cycles described by

$$X = A\cos(\omega t + \delta)$$

amplitude phase

Complex Numbers

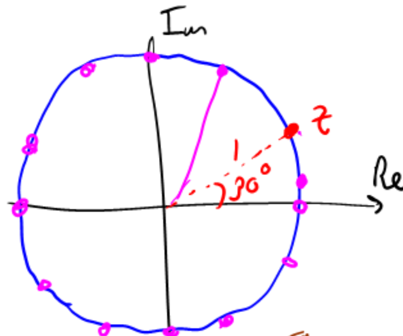
$\mathbb{C}$



Lots of natural cycles

Ex: repeated powers of $z = \frac{\sqrt{3}}{2} + \frac{1}{2}i$

z^n	$a + ib$	(r, θ)
z	$\sqrt{3}/2 + 1/2\, i$	$(1, 30°)$
z^2	$1/2 + \sqrt{3}/2\, i$	$(1, 60°)$
z^3	i	$(1, 90°)$
z^4	$-1/2 + \sqrt{3}/2\, i$	$(1, 120°)$
z^5	$-\sqrt{3}/2 + 1/2\, i$	$(1, 150°)$
z^6	-1	$(1, 180°)$
z^7	$-\sqrt{3}/2 - 1/2\, i$	$(1, 210°)$
z^8	$-1/2 - \sqrt{3}/2\, i$	$(1, 240°)$
z^9	$-i$	$(1, 270°)$



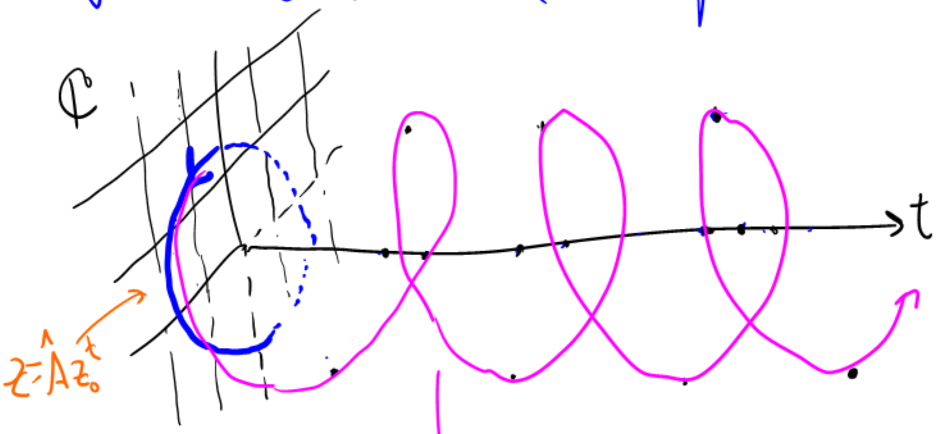
Continuous Cycles:

$$z = \hat{A}\, z_0^{\,t}$$

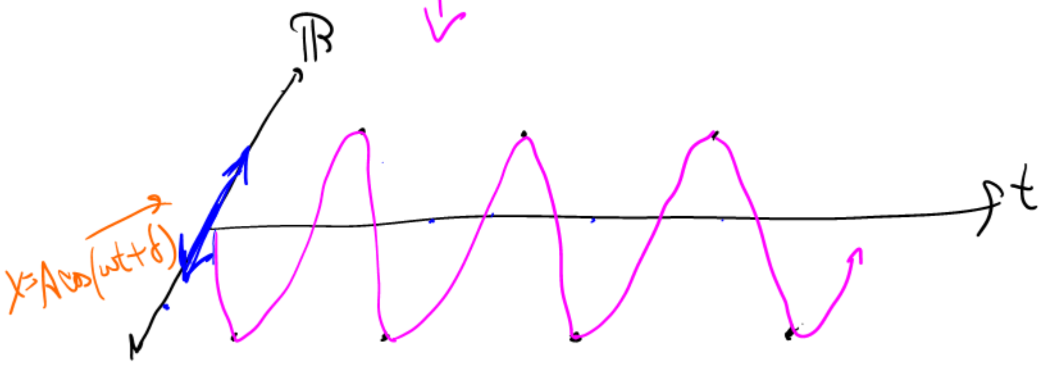
z^{10}	$1/2 - \sqrt{3}/2\, i$	$(1, 300°)$
z^{11}	$\sqrt{3}/2 - 1/2\, i$	$(1, 330°)$
z^{12}	1	$(1, 360°) = (1, 0°)$
z^{13}	repeat $(=z)$	

complex $(1, \omega)$
amplitude (A, δ)

Visualization of complex-valued wave $z = \hat{A} z_0^t$



Visualization of Real-valued wave $x = A\cos(\omega t + \delta)$



Combining real-valued waves

$$A_1 \cos(\omega t + \delta_1) + A_2 \cos(\omega t + \delta_2)$$

$$= A_1(\cos\omega t \cos\delta_1 - \sin\omega t \sin\delta_1)$$
$$+ A_2(\cos\omega t \cos\delta_2 - \sin\omega t \sin\delta_2)$$

$$= (A_1\cos\delta_1 + A_2\cos\delta_2)\cos\omega t$$
$$- (A_1\sin\delta_1 + A_2\sin\delta_2)\sin\omega t$$

$$= A\cos(\omega t + \delta)$$

where

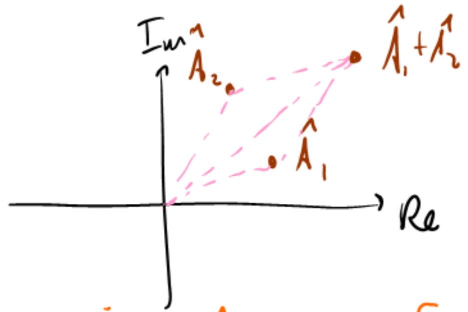
$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\delta_1 - \delta_2)}$$

$$\delta = \tan^{-1}\left(\frac{A_1\sin\delta_1 + A_2\sin\delta_2}{A_1\cos\delta_1 + A_2\cos\delta_2}\right)$$

Combining complex-valued waves

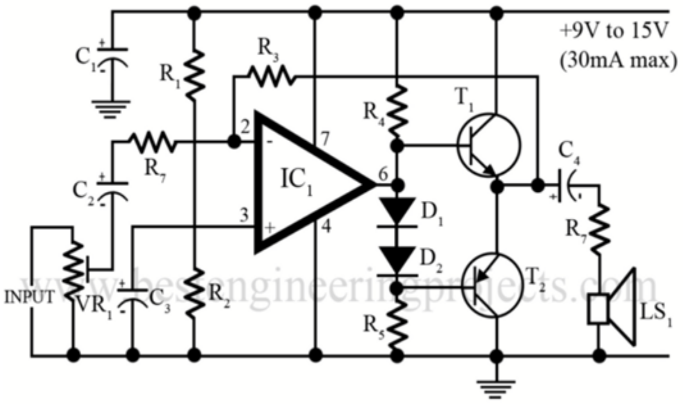
$$\hat{A}_1 z_0^t + \hat{A}_2 z_0^t$$

$$= (\hat{A}_1 + \hat{A}_2) z_0^t$$



$\hat{A}_1 = A_1 e^{i\delta_1}$ $\quad \hat{A}_2 = A_2 e^{i\delta_2}$

$z_0 = e^{i\gamma}$


+9V to 15V
(30mA max)
C1
R1
R3
R4
T1
C4
R7
C2
IC1
D1
D2
T2
INPUT
VR1
C3
R2
R5
LS1
2
7
6
3
4

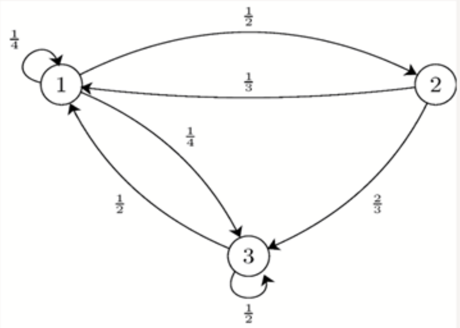


Figure 11.7 - A state transition diagram.

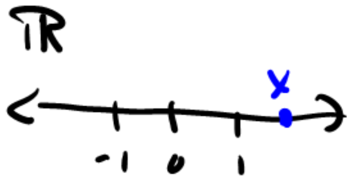
$$P = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$

A historical wrong:

Terrible mathematical terminology

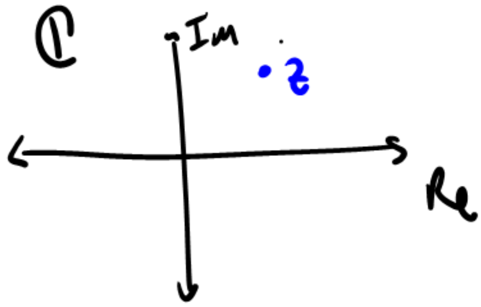
The ~~Real~~ Numbers

Linear

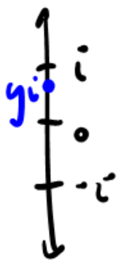


The ~~Complex~~ Numbers

Planar



Imaginary Numbers



If one formerly contemplated this subject from a false point of view and therefore found a mysterious darkness, this is in large part attributable to clumsy terminology. Had one not called +1, −1, $\sqrt{-1}$ positive, negative, or imaginary (or even impossible) units, but instead, say, direct, inverse, or lateral units, then there could scarcely have been talk of such darkness. — Gauss (1831)[30](p 638)[29]

Wikipedia
Complex Numbers