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GROUP-ALGEBRAIC CHARACTERIZATION OF SPIN PARTICLES: SEMI-SIMPLICITY, SO(2N) STRUCTURE AND IWASAWA DECOMPOSITION

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Outline

- Doran in 1993 showed that every linear transformation can be represented as a monomial of vectors in geometric algebra, every Lie algebra as a bivector algebra, and every Lie group as a spin group.
- Schwinger's realization of $su(1,1)$ Lie algebra with creation and annihilation operators was defined with spatial reference in the Pauli matrix representation [W. Pauli, 1927; J. Schwinger, 1945].
- Several relations as well as connections were observed in spin particles such as fermionic, bosonic, parastatistic Lie algebras, and in geometric algebras such as the Clifford algebra, Grassmannian algebra and so on [G. Sobezyk, 2015].
- Sobczyk proved that the spin half particles can be represented by geometric algebras. [G. Sobezyk, 2015]
- T.D. Palev in 1976 highlighted that a semi-simple Lie algebra can be generated by the creation and annihilation operators. In all the above mentioned works, the classical groups such as B_n and D_n play a crucial role in the spin particle Lie algebra.

- Moreover, several evidences from particle and theoretical physics showed the connection between quantum spin particle Lie algebra and Clifford algebra [Doran et *al.*, 1993].
- The spin of elementary particles obeying the Fermi-Dirac statistics, the Bose-Einstein statistics, the quantization of parastatistics such as parafermions and parabosons also gained much attention in the literature[Pauli Jr, 1927; Biedenharn and J.Louck, 1981; Doran et *al.*; Schwinger, 1945;Thankappan,1972 Sobezyk, 2015].
- In the opposite, exhaustive investigations on spin particle creation and annihilation and their angular momentum in connection with Lie groups, Lie algebras, Clifford algebras, and their representations are still lacking. This study aims at fulfilling this gap. The Iwasawa decomposition, introduced by the Japanese mathematician Kenkichi Iwasawa, generalizes the Gram-Schmidt orthogonality process in linear algebra[Holman III and Biedenharn Jr, 1966].

Motivated by all the above mentioned works, a natural questions arise:

1. Is spin in Mathematics the same as the spin in particle Physics?
2. If spin (n) is a double cover of SO (n) group, what is the cover of spin (2) and what is also the cover for spin (1/2) are they both related?
3. Is it possible to construct the Iwasawa decomposition at both the Lie algebra and Lie group levels of the spin particles ?

But before dealing with the main results, and as a matter of clarity in the development, let us briefly recall the main definitions, the known results, and the appropriate notations useful in the sequel.

Definition 1 [Green, 1953]

Let $a_1^{\pm}, ..., a_n^{\pm}$ be the creation and annihilation operators for a system consisting of n-fermions with commutator relations :

$$[a_i^-, a_j^+] = \delta_{ij} \tag{3.1}$$

$$[a_i^-, a_j^-] = [a_i^+, a_j^+] = 0, \tag{3.2}$$

or, of n-parafermions with

$$[[a_i^-, a_i^+], a_j^{\pm}] = \pm 2\delta_{ij} a_j^+, \tag{3.3}$$

where

$$[X, Y] := XY - YX. \tag{3.4}$$

Let T be the associative free algebra of a_i, a_j; $i, j \in N = \{1, 2, ...n\}$, and I be the two sided ideal in T generated by the relation (**??**). The Quotient (factor algebra)

$$Q = \frac{T}{I} \tag{3.5}$$

is called para-Fermi algebra, for all $X, Y \in Q$. This is an infinite dimensional Lie algebra with respect to the bracket defined by the equation (**??**).

Definition 2 [Palev, 1976]

Let g be a semi-simple Lie algebra generated by n pairs $a_1^{\pm}, ..., a_n^{\pm}$ of creation and annihilation operators. The elements

$$h_i = \frac{1}{2}[a_i^-, a_i^+], i = 1, ...n \tag{3.6}$$

are contained in a Cartan subalgebra H of g. The rank of $g \geq n$. If the semi-simple Lie algebra g of rank n is generated by n pairs of creation and annihilation operators, then, with respect to the basis of the Cartan subalgebra, the creation (resp. annihilation) operators are negative (resp. positive) root vectors. The correspondence with their roots is:

$$a_i^{\pm} \longleftrightarrow \pm h^{*i}. \tag{3.7}$$

where $\pm h^{*i}$ is a basis in the space dual to the Cartan subalgebra.

Definitions 3

Let now m be an n-dimensional oriented real vector space with an inner product $<,>$.

- We define the Clifford algebra [La Harpe, 1972] $Cl(m)$ over m by the quotient $T(m)/I$, where $T(m)$ is a tensor algebra over m and I is the ideal generated by all elements $v \otimes v + < v, v > 1$, $v \in m$.
- The multiplication of $Cl(m)$ will be denoted by $x \cdot y$.
- Let $p : T(m) \longrightarrow Cl(m)$ be the canonical projection. Then, $Cl(m)$ is decomposed into the direct sum $Cl^+(m) \oplus Cl^-(m)$ of the p-images of the elements of even and odd degrees of $T(m)$, and m is identified with the subspace of $Cl(m)$ through the projection p.
- Let $e_1, e_2, \cdots, e_n$ be an oriented orthonormal basis of m. The map: $e_{i2} \cdot e_{i2} \cdot \cdots \cdot e_{ip} \mapsto (-1)^p e_{ip} \cdot \cdots e_{i2} \cdot e_{i1}$ defines a linear map of $Cl(m)$ and the image of $x \in Cl(m)$ by this linear map is denoted by $\bar{x}$.

- The spin group is defined by:

$$\mathrm{Spin}(V) = \{x \in C\ell^{+}(V) : xVx^{-1} \subset V \text{ and } x\bar{x} = 1\}.$$

- Let $C\ell(V) = C\ell_{p,q} = C\ell(\mathbb{R}_{p,q}) = \mathbb{R}_{p,q}$ be the Clifford algebra over $\mathbb{R}$. Consider $\mathbb{R}^{*}_{p,q} = C\ell^{*}_{p,q}$ the group of invertible elements of $\mathbb{R}_{p,q}$. The exponential of $y \in C\ell_{p,q}$ is defined by:

$$\exp(y) = \sum_{n=0}^{\infty} \frac{1}{n!} y^{n}.$$

- Let $\pi : \mathrm{Spin}(V) \to \mathrm{SO}(V)$ be defined by $\pi(x)v = xvx^{-1}$. Then, the differential $\dot{\pi}$ of π is given by: $\dot{\pi}(x)v = xv - vx, \quad$ for $x \in \mathfrak{spin}(V)$ and $v \in V$.

Suppose now M is an oriented Riemannian manifold.

Definition 4[Milnor; Rodrigues]

A spin structure on M consists of a principal fiber bundle $\pi : P_{\mathrm{Spin}^e_{p,q}}(M) \longrightarrow M$ with a group $P_{\mathrm{Spin}^e_{p,q}}$ and the fundamental map (two fold cover)

$$s : P_{\mathrm{Spin}^e_{p,q}}(M) \longrightarrow P_{\mathrm{SO}^e_{p,q}}(M),$$

satisfying the following conditions:

(i) $\pi(s(p)) = \pi_s(p)$ for every $p \in P_{\mathrm{Spin}^e_{p,q}}(M)$; π is the projection map of the bundle $P_{\mathrm{SO}^e_{p,q}}(M)$.

(ii) $s(pu) = s(p)Ad_u$ for every $p \in P_{\mathrm{Spin}^e_{p,q}}(M)$ and

$$Ad : \mathrm{Spin}^e_{p,q} \to Aut(C\ell_{p,q}), \quad Ad_u : \mathbb{R}_{p,q} \mapsto uxu^{-1} \in C\ell_{p,q}.$$

the following diagram must commute

$$\begin{array}{ccc} P_{\mathrm{Spin}^e_{p,q}}(M) & \xrightarrow{\quad s \quad} & P_{\mathrm{SO}^e_{p,q}}(M) \\ & \searrow^{\pi_s} \quad \swarrow_{\pi} & \\ & M & \end{array}$$

Definitions 5 [Milnor; Rodrigues]

- A spin manifold is an orientable manifold M together with a spin structure on the tangent bundle of M.
- A spin group is a compact dimensional Lie group.

Spin Lie group

The Lie algebra $\mathfrak{spin}(j)$ of spin particles can be represented by classical matrices, which makes it easier to see their algebraic nature:

$$\mathfrak{spin}(j) = \begin{cases} \text{higgs} & j = 0; \\ \text{fermions} & j = \frac{\mathbb{Z}}{2} \text{ when odd integer spins are considered;} \\ \text{bosons} & j = \mathbb{Z} \text{ when positive integer spins are considered.} \end{cases}$$

The Lie algebra $\mathfrak{sl}(2n, \mathbb{C})$ can represent the fermion spin Lie algebra of elementary particles in quantum physics. As indicated in the mapping below. We define $\frac{\mathbb{Z}}{2}$ as fraction of the form $(\frac{2k+1}{2},\ k = 0, 1, 2, 3, \ldots)$:

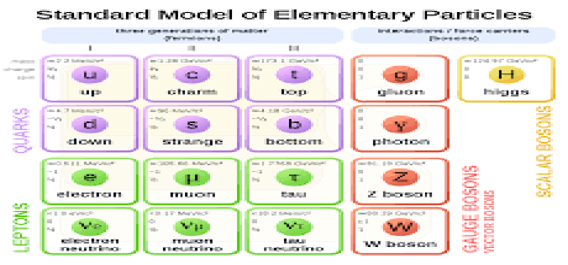
$$\mathfrak{sl}(2n, \mathbb{C}) \longrightarrow \mathfrak{spin}\left(\frac{\mathbb{Z}}{2}\right) \longrightarrow \text{fermions.}$$

- The Lie group $\mathrm{SL}(2n, \mathbb{C})$ structure can represent the fermion Spin Lie group analogue:

$$\mathrm{SL}(2n, \mathbb{C}) \longrightarrow \mathrm{Spin}(\tfrac{\mathbb{Z}}{2}) \longrightarrow \text{fermions},$$

while the Lie group $\mathrm{SL}(2n+1, \mathbb{C})$ represents the boson Spin Lie group analogue:

$$\mathrm{SL}(2n+1, \mathbb{C}) \longrightarrow \mathrm{Spin}(\mathbb{Z}) \longrightarrow \text{bosons}.$$



We seek a transformation to a set basis denoted $|SM\rangle$, which obeys [Holman III and Biedenharn Jr, 1966; Thankappan, 1972; Biedenharn and Louck, 1981]:

$$S^2|SM\rangle = S(S+1)\hbar^2|SM\rangle, \tag{3.8}$$

$$S_z|SM\rangle = M\hbar|SM\rangle, \tag{3.9}$$

$$S_\pm|SM\rangle = \hbar\sqrt{S(S+1) - M(M\pm 1)}|S, M\pm 1\rangle. \tag{3.10}$$

- In relation to unitary transformation

$$|SM\rangle = \sum_{m_1 m_2} U^{s_1 s_2}_{m_1 m_2; sm}|m_1 m_2\rangle, \tag{3.11}$$

- where $U^{s_1 s_2}_{i,j}$ is the ij^{th} element of the unitary matrix $U^{s_1 s_2}$ that transforms the basis $|m_1 m_2\rangle$ to the basis $|SM\rangle$ [Thankappan, 1972].
- Using the closure property of the basis $|m_1 m_2\rangle$,

$$|SM\rangle = \sum_{m_1 m_2} |s_1 s_2 m_1 m_2\rangle\langle s_1 s_2 m_1 m_2|SM\rangle, \tag{3.12}$$

- comparing equation (**??**) and (**??**) lead to:

$$U^{s_1 s_2}_{m_1 m_2 ; sm} \equiv \langle s_1 s_2 m_1 m_2 | sm \rangle . \tag{3.13}$$

- The Clebsch-Gordan coefficients (C.G) are obtained as:

$$U^{s_1 s_2}_{m_1 m_2 ; sm} := C^{s_1 s_2 s}_{m_1 m_2 m} . \tag{3.14}$$

- The Wigner coefficients of the SU(2) group are then derived as:

$$C^{s_1 s_2 S}_{m_1 m_2 M} = [2S+1]^{\frac{1}{2}} (-1)^{s_2 + m_2} \left[\frac{(S+s_1-s_2)!(S-s_1+s_2)!(s_1+s_2-S)!}{(S+s_1+s_2+1)!(s_1-m_1)!(s_1+m_1)!} \right. \tag{3.15}$$

$$\left. \times \frac{(S+M)!(S-M)!}{(s_2+m_2)!(s_2-m_2)!} \right]^{\frac{1}{2}} \sum_k (-1)^k \tag{3.16}$$

$$\times \frac{(S+s_2+m_1-k)!(s_1-m_1+k)!}{k!(S-s_1+s_2-k)!(M+S+k)!(s_1-s_2-M+K)} . \tag{3.17}$$

Lemma I

Any spin particle Lie algebra admits a Clifford algebra and a spin group structure.

Proof.

Consider any $\mathfrak{spin}(j)$ with $j = 0, \frac{1}{2}, 1, \ldots,$ satisfying the spin particle commutator and anticommutator relations (**??**), (**??**), (**??**) as well as the spin Lie algebra commutation bracket rule. It is obvious that the Lie algebra $\mathfrak{spin}(j)$ is a Clifford algebra. Thus, the $\mathfrak{spin}(j)$ exponential is just

$$\exp : \mathfrak{spin}(j) \rightarrow \mathrm{Spin}(J)$$

where $\mathrm{Spin}(J)$ is the spin group. Hence, any spin particle admits a spin group. □

Lemma II

Any spin group of a spin particle admits an almost complex spin manifold (Riemannian manifold) and a spin Lie group structure.

Proof.

From Lemma **??**, any spin particle admits a spin group. Also, from Definition **??**, the spin group, say Spin(J), has a group structure with an almost complex manifold. Thus, from Definition 3.3, the spin particle, say Spin(J) with $J = 0, \frac{1}{2}, \ldots,$ admits a spin manifold. Next, we see that any spin particle has a spin group, say Spin(J). Since any spin particle has a spin manifold, we observe that Spin(J) is a spin group and, hence, a spin Lie group. □

Proposition I

Any spin half odd integer (resp. integer spin) Lie group is a fourfold cover of the compact Lie group $\mathrm{SO}(2n)$ (resp. a double cover of $\mathrm{SO}(2n+1)$).

Proof

The fermion quantum structure can be given as:

$$\mathrm{Spin}(J) \longrightarrow \mathrm{Spin}(\tfrac{\mathbb{Z}}{2}) \ .$$

The map

$$\mathrm{SL}(2n,\mathbb{C}) \longrightarrow \mathrm{Spin}(\tfrac{\mathbb{Z}}{2}) \longrightarrow \mathrm{SO}(2n),$$

where $\mathrm{SO}(2n)$ conserves the quadratic form in $\mathbb{C}^{2n}$.

Proof

where $\mathrm{SO}(2n)$ conserves the quadratic form in $\mathbb{C}^{2n}$. The compact simple Lie group $\mathrm{SO}(2n)$ is fourfold connected and its center $Z(\mathrm{G})$ is $\mathbb{Z}_4$ when n is odd or $\mathbb{Z}_2 \times \mathbb{Z}_2$ when n is even. Since $\mathrm{Spin}(\frac{\mathbb{Z}}{2})$ is a fermion with $\mathbb{Z}$ as an odd integer, the diagram

$$\begin{array}{ccccc} \mathrm{SL}(2n,\mathbb{C}) & \longrightarrow & \mathrm{Spin}(\frac{\mathbb{Z}}{2}) & & \\ & & & \searrow & \\ \uparrow & & \downarrow & & \text{fermions} \\ & & & \nearrow & \\ \mathrm{SU}^*(2n) & \longrightarrow & \mathrm{SO}(2n) & & \end{array}$$

must commute. Thus, the fermion Spin Lie group is a fourfold cover of $\mathrm{SO}(2n)$. Similarly, the boson quantum structure can be given as:

$$\mathrm{Spin}(J) \longrightarrow \mathrm{Spin}(\mathbb{Z})$$

where $\mathbb{Z}$ is an integer. The map

$$\mathrm{SL}(2n+1,\mathbb{C}) \longrightarrow \mathrm{Spin}(\mathbb{Z}) \longrightarrow \mathrm{SO}(2n+1),$$

where $\mathrm{SO}(2n+1)$ conserves the quadratic form in $\mathbb{C}^{2n+1}$. The compact simple Lie group $\mathrm{SO}(2n+1)$ is doubly connected and its center $Z(\mathrm{G})$ is $\mathbb{Z}_2$. Since $\mathrm{Spin}(\mathbb{Z})$ is a boson, where $\mathbb{Z}$ is an integer, the diagram

$$\begin{array}{ccccc} \mathrm{SL}(2n+1,\mathbb{C}) & \longrightarrow & \mathrm{Spin}(\mathbb{Z}) & & \\ & \searrow & \downarrow & \searrow & \\ & & \mathrm{SO}(2n+1) & \longrightarrow & \text{boson} \end{array}$$

must commute. We conclude that the boson spin Lie group is the double cover of $\mathrm{SO}(2n+1)$. See [Helgason, 1962] for more details.

Theorem I

Any spin Lie group $\mathrm{Spin}(J)$ of a spin particle is:

(i) connected;

(ii) semi-simple if and only if its simple roots are one of the Dynkin's root systems $\Pi(B_n)$ or $\Pi(D_n)$ associated with the classical groups $\mathrm{SO}(2n+1)$ and $\mathrm{SO}(2n)$, respectively.

Proof

We let $\mathrm{Spin}(J)$ be a spin Lie group with $J = 0, \frac{1}{2}, 1, \ldots$. For $\mathrm{Spin}(0)$, $\mathrm{Spin}(\frac{1}{2})$, and $\mathrm{Spin}(1)$, we have, respectively, the diagram:

$$\begin{array}{ccccc} \mathrm{SL}(1,\mathbb{C}) & \longrightarrow & \mathrm{Spin}(0) & & \\ & \searrow & \downarrow & \searrow & \\ & & \mathrm{SO}(1) & \longrightarrow & \text{boson (Higgs)}, \end{array}$$

$$\begin{array}{ccccc} \mathrm{SL}(2,\mathbb{C}) & \longrightarrow & \mathrm{Spin}(\tfrac{1}{2}) & & \\ & \searrow & \downarrow & \searrow & \\ & & \mathrm{SO}(2) & \longrightarrow & \text{fermion}, \end{array}$$

$$\begin{array}{ccccc} \mathrm{SL}(3,\mathbb{C}) & \longrightarrow & \mathrm{Spin}(1) & & \\ & \searrow & \downarrow & \searrow & \\ & & \mathrm{SO}(3) & \longrightarrow & \text{boson}. \end{array}$$

these Lie groups $\mathrm{SO}(1)$, $\mathrm{SO}(2)$ and $\mathrm{SO}(3)$ are connected [Helgason, 1962].

The results can be extended to all spin Lie groups of elementary particles as shown in Proposition **??**. Fermions and bosons spin Lie groups are fourfold connected and double connected, respectively. More specifically, the spin Lie groups such as Spin($\frac{1}{2}$) fourfold covers the compact Lie group $\mathrm{SO}(2)$, while Spin(1) double covers $\mathrm{SO}(3)$.
[Proof of (ii)] We know that the creation and annihilation operators generate a semi-simple Lie algebra $\mathfrak{g}$ of rank n which is a direct sum of classical Lie algebras

$$\mathfrak{g} = B_{m_1} \oplus \cdots \oplus B_{m_k},$$

where $m_1 + \cdots + m_k = n$. Therefore, the creation and annihilation operators of spin particles generate simple Lie algebra $\mathfrak{g}$ of rank n isomorphic to the classical algebra B_n with a complete system Φ of roots orthogonal with respect to the Killing form [Palev, 1976]. Also, from equations (**??**) and (**??**), when we compare the bracket relation to that of the Dynkin's root $\sum D_n$, see equation (**??**), we observe that there is a correspondence.
In Lemma **??**, we showed that every spin group of a spin particle is a spin Lie group.

We can determine the system of simple roots $\Pi(B_n)$ and $\Pi(D_n)$ associated with the classical groups $\mathrm{SO}(2n)$ and $\mathrm{SO}(2n+1)$. A semi-simple Lie group G is completely determined by the system $\Pi(\mathrm{G})$ of its simple roots [?]. Thus, the spin Lie group of a spin particle is completely determined by $\Pi(\mathrm{G})$ of its simple roots. The converse is trivial since the classical groups [?, ?] $\mathrm{SO}(2n)$ and $\mathrm{SO}(2n+1)$ correspond with the $\Pi(B_n)$ and $\Pi(D_n)$ (Dynkin's root system), which are the operators of the quantum spin particles generated by the creation and annihilation operators of rank n, since the spin Lie group is connected and its Lie algebra is semi-simple. Thus, the spin Lie group is semi-simple. $\square$

The $sl(2,\mathbb{C})$ Lie algebra can be decomposed into the compact real $su(2)$ and imaginary $isu(2)$ forms, or $sl(2,\mathbb{R})$ and $isl(2,\mathbb{R})$. It is only natural to seek the real form of the spin half particle Lie algebra in terms of Pauli matrices [Pauli Jr, 1927], which are $sl(2,\mathbb{C})$ matrix basis elements.

Proposition II

The real Lie algebra spin $\left(\frac{1}{2}\right)$ of spin half particles (Spin $\left(\frac{1}{2}\right)$) is given by spin $\left(\frac{1}{2}\right) = \{S \in M_2(\mathbb{R}) | TrS = 0\}$.

1. The elements $S_k = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $S_+ = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ form a basis of the spin $\left(\frac{1}{2}\right)$.
2. The commutation relations are given by : $[S_k, S_z] = -\hbar S_x, [S_k, S_+] = \hbar S_z, [S_z, S_+] = \hbar S_+$.

Proof.

Take an arbitrary angular momentum $\mathfrak{spin}(\frac{1}{2})$ with spinors $\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\chi_{\frac{1}{2}} + b\chi_{-\frac{1}{2}}$, $\chi_{\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\chi_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Let

$$S_x = \frac{S_+ + S_-}{2} \quad and \quad S_y = \frac{S_+ - S_-}{2i}.$$

From the above equations (**??**) and (**??**), we can write S^2 and S_z in terms of spinors. Indeed,

$$S^2\chi_{\frac{1}{2}} = \hbar^2\frac{1}{2}\left(\frac{1}{2}+1\right)\left|\chi_{\frac{1}{2}}\right\rangle = \frac{3}{4}\hbar^2\chi_{\frac{1}{2}}, \quad S^2\chi_{-\frac{1}{2}} = \hbar^2\frac{3}{4}\chi_{-\frac{1}{2}}. \tag{5.1}$$

From equations (**??**), we can deduce

$$S^2 = \frac{3}{4}\hbar^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{3}{4}\hbar^2 I,$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity matrix. Similarly,

$$S_z\chi_{\frac{1}{2}} = \frac{\hbar}{2}\chi_{\frac{1}{2}} \text{ and } S_z\chi_{-\frac{1}{2}} = -\frac{\hbar}{2}\chi_{-\frac{1}{2}}.$$

Therefore,

$$S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\hbar}{2}\sigma_z.$$

By analogous computations, we get:

$$S_+ = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \hbar\sigma_+ \tag{5.2}$$

and

$$S_- = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \hbar\sigma_-. \tag{5.3}$$

Similarly,

$$S_x = \frac{S_+ + S_-}{2} = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar}{2}\sigma_x \tag{5.4}$$

and

$$S_y = \frac{S_+ - S_-}{2i} = -iS_k = \frac{-i\hbar}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \frac{-i\hbar}{2}\sigma_k = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{\hbar}{2}\sigma_y. \tag{5.5}$$

Lemma III

For a $\mathfrak{spin}(\frac{1}{2})$ there exists an orthogonal (skew symmetric) matrix element S_k (with $\hbar = 1$), which can be transformed into an $\mathrm{SO}(2)$ compact Lie group. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}) = \mathrm{G}$, the stabilizer of $i \in \mathbb{C}$ under the action of g is the subgroup $K = \mathrm{SO}(2)$.

Remark

The $\mathfrak{spin}\left(\frac{1}{2}\right) \in sl(2, \mathbb{C})$. Thus, it is complex, and for good notation, we write $\mathfrak{spin}\left(\frac{1}{2}, \mathbb{C}\right) \subset \mathfrak{sl}(2, \mathbb{C})$. For the real form, we write $\mathfrak{spin}_{\mathbb{R}}\left(\frac{1}{2}\right) \subset \mathfrak{sl}(2, \mathbb{R})$. Finally,

$$\mathfrak{spin}\left(\frac{1}{2}, \mathbb{C}\right) = \mathfrak{spin}_{\mathbb{R}}\left(\frac{1}{2}\right) \oplus i\mathfrak{spin}_{\mathbb{R}}\left(\frac{1}{2}\right).$$

For simplicity, in the next section, we use the usual notation $\mathfrak{spin}_{\mathbb{R}}(\frac{1}{2})$ to be the real form $\mathfrak{spin}\left(\frac{1}{2}, \mathbb{R}\right)$ of the spin half particle. Note, when $\hbar = 1$, $\mathfrak{spin}_{\mathbb{R}}(\frac{1}{2}) = \mathfrak{su}(2)$ and $\mathfrak{spin}\left(\frac{1}{2}, \mathbb{C}\right) \subset \mathfrak{sl}(2, \mathbb{C})$.

Theorem II

(i) Let θ, t, ξ be arbitrary real numbers, and put $\hbar k_\theta = \exp(\theta S_k)$, $\hbar d_t^{\frac{1}{2}} = \exp(tS_z)$, and $\hbar n_\xi = \exp(\xi S_+)$. Then, the subgroups $\hbar^3 KDN$ of Spin $(\frac{1}{2})$ are defined by: $\hbar K_\theta = \{\hbar k_\theta | \theta \in R\}$, $\hbar D = \{\hbar d_t^{\frac{1}{2}} | t \in R\}$ and $\hbar N = \{\hbar n_\xi | \xi \in R\}$. We have:

$$\hbar k_\theta = \begin{pmatrix} \hbar\cos\frac{\theta}{2} & \hbar\sin\frac{\theta}{2} \\ -\hbar\sin\frac{\theta}{2} & \hbar\cos\frac{\theta}{2} \end{pmatrix},\ \hbar d_t^{\frac{1}{2}} = \begin{pmatrix} \hbar e^{\frac{t}{2}} & 0 \\ 0 & \hbar e^{-\frac{t}{2}} \end{pmatrix},$$

$$\hbar n_\xi = \begin{pmatrix} \hbar & \hbar\xi \\ 0 & \hbar \end{pmatrix},$$

$\hbar K \cong \frac{\mathbb{R}}{4\pi\mathbb{Z}} \cong T$, $\hbar D \cong \mathbb{R}$, $\hbar N \cong \mathbb{R}$.

(ii) Any spin $(\frac{1}{2})$ particle is uniquely decomposable in the form:

$$spin\left(\frac{1}{2}\right) = \hbar^3\ k_\theta d_t^{\frac{1}{2}} n_\xi = \exp\left(\theta\langle{}^s_m|S_k|{}^s_m\rangle\right)\cdot\exp\left(t\langle{}^s_m|S_z|{}^s_m\rangle\right)\cdot\exp\left(\xi\langle{}^s_m|S_+|{}^s_m\rangle\right). \tag{6.1}$$

Theorem II: Iwasawa Decomposition of Spin $(\frac{1}{2})$ particle

If $spin\ (\frac{1}{2}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathrm{R})$, then, θ, t, ξ in Theorem II (i) are given by the relations:

$$\exp\left(i\frac{\theta}{2}\right) = \frac{a - ic}{\hbar^3\sqrt{a^2 + c^2}}, \tag{6.2}$$

$$\exp(t) = \frac{a^2 + c^2}{\hbar^6}, \tag{6.3}$$

and

$$\xi = \frac{\hbar^6(ab + cd)}{a^2 + c^2}. \tag{6.4}$$

Proof.

Since

$$\hbar k_\theta = \exp(\theta S_k) = \exp\left(\theta \langle {}^s_m | S_k | {}^s_m \rangle\right) \tag{6.5}$$

$$= \exp\left(\theta \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = \hbar \exp\left(\frac{\theta}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \tag{6.6}$$

and

$$\begin{aligned} \hbar K_\theta &= \hbar\left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{\theta}{2}\right)^{2n} \cdot I + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{\theta}{2}\right)^{2n+1} \cdot \sigma_k\right] \\ &= \hbar \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}. \end{aligned}$$

By isomorphism $\theta \longmapsto \hbar K_\theta$, we obtain: $\hbar K \cong \dfrac{R}{4\pi\mathbb{Z}} \cong T$.

Moreover,

$$\hbar d_t^{\frac{1}{2}} = \exp\left(t(S_z)\right) = \exp\left(t\langle{}^{s}_{m}|S_z|{}^{s}_{m}\rangle\right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}(t\, S_z)^n = \hbar \sum_{n=0}^{\infty} \frac{1}{n!}(t\, \sigma_z)^n = \hbar \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix}.$$

By isomorphism $t \longmapsto \hbar d_t^{\frac{1}{2}}$, we also have: $D \cong \mathbb{R}$. Now, since $(S_+)^2 = 0$,

$$\hbar n_\xi = \exp\left(\xi S_+\right) = \hbar \exp\left(\xi\, \sigma_+\right) = \exp\left(\xi\langle{}^{s}_{m}|S_+|{}^{s}_{m}\rangle\right) = \hbar \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}.$$

By matrix multiplication, we have:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \hbar^3\, k_\theta d_t^{\frac{1}{2}} n_\xi = \exp\left(\theta\langle{}^{s}_{m}|S_k|{}^{s}_{m}\rangle\right) \cdot \exp\left(t\langle{}^{s}_{m}|S_z|{}^{s}_{m}\rangle\right) \cdot \exp\left(\xi\langle{}^{s}_{m}|S_+|{}^{s}_{m}\rangle\right)$$

$$= \begin{pmatrix} \hbar^3 \exp\left(\frac{t}{2}\right) cos\frac{\theta}{2} & \hbar^3 cos\frac{\theta}{2} \exp\left(\frac{t}{2}\right) \xi + \hbar^3 sin\frac{\theta}{2} \exp\left(-\frac{t}{2}\right) \\ -\hbar^3 \exp\left(\frac{t}{2}\right) sin\frac{\theta}{2} & -\hbar^3 sin\frac{\theta}{2} \exp\left(\frac{t}{2}\right) \xi + \hbar^3 cos\frac{\theta}{2} \exp\left(-\frac{t}{2}\right) \end{pmatrix}.$$

yielding

$$a = \hbar^3 \exp\left(\frac{t}{2}\right) cos\frac{\theta}{2}, \quad c = -\hbar^3 \exp\left(\frac{t}{2}\right) sin\frac{\theta}{2},$$

and

$$a - ic = \hbar^3 \exp\left(\frac{t}{2} + i\frac{\theta}{2}\right).$$

Hence, $|a - ic| = \hbar^3 \exp\left(\frac{t}{2}\right)$, we easily get equations (**??**) and (**??**), and

$$ab + cd = \exp(t)\xi \tag{6.7}$$

from which we can clearly obtain equation (**??**). End of the proof.

Now we know that Spin $\left(\frac{1}{2}\right)$ is spanned by two states: $\{|\frac{1}{2}\ \frac{1}{2}\rangle, |\frac{1}{2}\ , -\frac{1}{2}\rangle\}$. From equations (**??**), (**??**) and (**??**), we can calculate the angular momentum for spin half integers such as $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$ and so on$\cdots$ [Thankappan, 1972].
A question arises: What can the general term (last term) of a spin half integer be? From a theoretical point of view, this can be useful in the study of particle rotational forms. We have the following results:

Theorem III

For any $\mathfrak{spin}(\frac{2n-1}{2})$ (fermions) quantum state, where $n = 1, 2, 3, \ldots,$ we have:

(i) $S^2 \left|S, M\right\rangle_n = \left(\frac{4n^2-1}{4}\right) \hbar^2 \left|S, M\right\rangle.$

(ii) $S_z \left|S, M\right\rangle_n = \pm\left(\frac{2n-k}{2}\right) \hbar \left|S, M\right\rangle$ where $k \leq 2n$, and $n = 1, 2, \ldots$, with $k = 1, 3, 5, \ldots$.

(iii) The n^{th} possible state of a spin half particle is given by:

$$M_{s_n} = 2S_n + 1 = 2n,$$

where $n = 1, 2, 3, \ldots$. The quantum state of the fermion is spanned by $2n$ states:

$$\left|\left(\frac{2n-1}{2}\right), \pm\left(\frac{2n-1}{2}\right)\right\rangle, \ldots, \left|\left(\frac{2n-1}{2}\right), \pm\left(\frac{2n-k}{2}\right)\right\rangle,$$

where $k = 1, 3, 5, \ldots,$ with $k \leq 2n$.

(iv) The ladder operators act as follows:

$$S_{+_n}\left|\left(\frac{2n-1}{2}\right)\ \left(\frac{2n-k}{2}\right)\right\rangle = \hbar\sqrt{(k-1)n - \left(\frac{(k-1)(k-1)}{4}\right)}\left|S, M+1\right\rangle, \tag{6.8}$$

$$S_{+_n}\left|\left(\frac{2n-1}{2}\right), -\left(\frac{2n-k}{2}\right)\right\rangle = \hbar\sqrt{(k+1)n - \left(\frac{(k+1)(k+1)}{4}\right)}\left|S, M{+}1\right\rangle, \tag{6.9}$$

Note

$$S_{y_n} = \frac{S_{+_n} - S_{-_n}}{2i} = \frac{\hbar\sigma_{k_n}}{2i} = -iS_{k_n}. \tag{6.10}$$

Theorem IV

For any $\mathfrak{spin}(\frac{2n-1}{2})$, the quantum state of the particle is spanned by $2n$ states and there exists orthogonal matrix element S_{k_n} in the S_{y_n} matrix which can be transformed into the classical group $\mathrm{SO}(2n)$ with natural numbers $n = 1, 2, 3, \ldots$. This compact Lie group $\mathrm{SO}(2n)$ corresponds to the Dynkin's root $\Pi(D_n)$.

Proof.

From Lemma III, From Lemma **??** we observe that the theorem is true for $n = 1$. For $\mathfrak{spin}(\frac{2n-1}{2})$ particle quantum state spanned by $2n$ states, we consider similar arguments for Theorem II, replacing the S_k matrix by the n^{th} matrix S_{k_n} and deducing in the same manner as in Lemma III to obtain the above Theorem III. Specifically, from Theorem II, there exists S_{k_n} matrix in the S_{y_n} matrix from equation 6.10. One can check that these matrices are orthogonal and generate $\mathrm{SO}(2n)$ with $n = 1, 2, 3 \ldots$. For $n = 1$ we have the compact Lie group $\mathrm{SO}(2)$ as in Lemma III. Finally, the correspondence to the Dynkin's roots $\Pi(D_n)$.

Proposition III [Sugiura, 1977]

For an element $g \in \mathrm{Spin}_{\mathbb{R}}\left(\frac{1}{2}\right)$ and $\theta \in \mathbb{R}$, let

$$gk_\theta = k_{g.\theta} d^{\frac{1}{2}}_{t(g,\theta)} n_{\xi(g,\theta)}$$

be the Iwasawa decomposition of gk_θ. If $\hbar = 1$, then, the following cocycle conditions hold, for g, g' which are projections in $\mathrm{SL}(2,\mathbb{R}) \supset \mathrm{Spin}_{\mathbb{R}}\left(\frac{1}{2}\right)$:

1. (i) $(gg') \cdot \theta \equiv g \cdot (g' \cdot \theta) \pmod{4\pi}$;
   (ii) $t(gg', \theta) = t(g, g' \cdot \theta) + t(g', \theta)$;
   (iii) $g \cdot (\theta + 2\pi) = g \cdot \theta + 2\pi \pmod{4\pi}$, $t(g, \theta + 2\pi) = t(g, \theta)$.

Proposition III [Sugiura, 1977]

1. If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then, we have:

(i) $\exp\left(\frac{i(g\cdot\theta)}{2}\right) = \dfrac{(a-ic)\cos\frac{\theta}{2} + (-b+id)\sin\frac{\theta}{2}}{|(a-ic)\cos\frac{\theta}{2} + (-b+id)\sin\frac{\theta}{2}|}$;

(ii) $\exp\left(t\left(g,\theta\right)\right) = |(a-ic)\cos\frac{\theta}{2} + (-b+id)\sin\frac{\theta}{2}|^2$;

(iii) $\xi\left(g,\theta\right) = \exp\left(-t\left(g,\theta\right)\right) \times \left[(ab+cd)\cos\theta + \frac{1}{2}\left(a^2 - b^2 + c^2 - d^2\right)\sin\theta\right]$;

(iv) $d\frac{(g\cdot\theta)}{d\theta} = \exp\left(-t\left(g,\theta\right)\right)$.

Theorem V[Main Result: Particle Decomposition]

Any spin Lie group G can be uniquely decomposed in the form:

$$\mathrm{G} = \text{Ӿ} K D^s N$$

where K is compact, D^s is a rotational function (d-function), and N is nilpotent (Ladder operators). We denote by Ӿ(α^{-1}) the fine structure constant and all other translational energy of elementary spin particles.

Fine structure constant

$$\alpha^2 = \frac{(k_c)(q)^2}{(h/\pi)(c)}, \quad \alpha^{-1} = 137.036...$$

$$\alpha = \frac{1}{137} = \text{Ӿ}.$$

Proof

We will give a physical interpretation to this theorem. Let us consider the Iwasawa decomposition of an electron which has a spin half. Suppose an electron is at rest in a homogeneous magnetic field. Its eigenfunctions do not depend upon its position. Given μ, the magnitude of Bohr magneton and H, the Hamiltonian, one then obtains the system of equations for $(\psi_\alpha, \psi_\beta)$ as follows

$$\mu[(H_x - iH_y)\psi_\beta + H_z\psi_\alpha] = E\psi_\alpha$$
$$\mu[(H_x + iH_y)\psi_\alpha - H_z\psi_\beta] = E\psi_\beta$$

from which we find that $E = \pm\mu|H|$. If we denote the angle between the field direction and the z-axis by θ and normalizes $(\psi_\alpha, \psi_\beta)$ by the way of

$$|\psi_\alpha|^2 + |\psi_\beta|^2 = 1. \tag{7.1}$$

Proof Cont'd

This then corresponds to the determinant of the compact (K) function in the Iwasawa decomposition, that is, $\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}=1$. If one suddenly rotates the external magnetic field in the z-direction; $\cos^2\frac{\theta}{2}$ is then the fraction of the electron with moments that are directed parallel to the z-axis and $\sin^2\frac{\theta}{2}$ is the fraction of electron with moments that are directed anti-parallel to the z-axis and vice versa. Consider the subgroup $D^s=\{D^s_{mm'}(t)\mid t\in R\}$, with

$$D^s_{mm'}(t)=d^s_{mm'}(t)=d^s_t.$$

D^s is the rotational function(d-function) and $D^s_{mm'}(t)$ is the Clebsch-Gordon coefficient. Given the identity

$$\exp(itK_y)K_z\exp(-itK_y)=\cosh t\cdot K_z-\frac{1}{2}\sinh t\cdot(K_++K_-) \tag{7.2}$$

which can be defined using the Campbell-Hausdorff formula together with the $SU(1,1)$ commutation relation.

Ptoof Cont'd

Taking the matrix element of

$$K_z \exp(-itK_y) = \cosh t \exp(-itK_y)K_z - \frac{1}{2}\sinh t \exp(-itK_y)(K_+ + K_-) \quad (7.3)$$

between $|s, m\rangle$ and $|s, m'\rangle$, one can easily obtain the recurrence formula for the d-function. For the finite $(2S+1)$-dimensional case, we have:

$$(m' - \cosh t \cdot m)d^s_{mm'}(t) + \frac{1}{2}\sinh t \left(-\sqrt{(s-m)(s+m+1)}d^s_{mm'}(t) + \sqrt{(s+m)(s-m+1)}d^s_{m'm-1}(t)\right) = 0 \quad (7.4)$$

Cont'd

which leads to the factorization of the d-function of $SU(1,1)$:

$$d^s_{m'm}(t)$$
$$= \left[\frac{\Gamma(s-m+1)\Gamma(s+m'+1)}{\Gamma(s+m+1)\Gamma(s-m'+1)}\right]^{\frac{1}{2}} \frac{1}{\Gamma(m'-m+1)} \times$$
$$\left(\cosh\left(\frac{t}{2}\right)\right)^{2s+m-m'} \left(\sinh\left(\frac{t}{2}\right)\right)^{m-m'} \times_2 F_1\left(m; \tanh^2\left(\frac{t}{2}\right)\right) \qquad (7.5)$$

where $m' \geq m$. Substituting formula (**??**) into (**??**), we obtain:

$$(s+m)(s-m+1)\tanh^2\left(\frac{t}{2}\right) \times_2 F_1\left(m-1; \tanh^2\left(\frac{t}{2}\right)\right)$$
$$+ \cosh^{-2}\left(\frac{t}{2}\right)(m' - \cosh t \cdot m) \times_2 F_1\left(m; \tanh^2\left(\frac{t}{2}\right)\right)$$
$$-(m'-m+1)(m'-m) \cdot \times_2 F_1\left(m+1; \tanh^2\left(\frac{t}{2}\right)\right) = 0. \qquad (7.6)$$

proof Cont'd

$$d^s_{m'm}(t) = \left[\frac{\Gamma(s-m+1)\Gamma(s+m+1)}{\Gamma(s-m'+1)\Gamma(s+m'+1)}\right]^{\frac{1}{2}} \left(\cosh\left(\frac{t}{2}\right)\right)^{-m-m'} \times \left(\sinh\left(\frac{t}{2}\right)\right)^{m-m'} P^{m'-m,-m-m'}_{s+m} \cosh(t). \tag{7.7}$$

We note that the form $D^s_{mm'} = d^s_{mm'}(t) = d^s_t$ and $d^{\frac{1}{2}}_t$ is the Abelian subgroup in the Iwasawa decomposition.
Finally, for the Nilpotent (N) function say, K_+ in the 2-dimensional non-unitary representation is the non-compact operator which generates elements of the parabolic subgroup of $SU(1,1)$.

Is Spin(n) in Mathematics the same as the Spin(J) in particle Physics?

In mathematics the spin group *spin* (n) is a double cover of the special orthogonal group SO (n).

$$\begin{array}{c} Spin\ (3) \\ \downarrow \\ SO(3) \end{array}$$

however, in particle Physics

$$\begin{array}{ccccc} SL(3,\mathbb{C}) & \longrightarrow & \text{Spin }(1) & & \\ & \searrow & \downarrow & \searrow & \\ & & SO(3) & \longrightarrow & boson, \end{array}$$

and similarly,In Mathematics

$$\begin{array}{c} \mathit{Spin}\ (2) \\ \downarrow \\ \mathrm{SO}(2) \end{array}$$

while in Physics;

$$\begin{array}{ccccc} \mathrm{SL}(2,\mathbb{C}) & \longrightarrow & \mathrm{Spin}\ \left(\frac{1}{2}\right) & & \\ & \searrow & \downarrow & \searrow & \\ & & \mathrm{SO}(2) & \longrightarrow & \mathit{fermion}. \end{array}$$

Remark

The Iwasawa decomposition of the spin half particle into compact, rotational (Abelian), and nilpotent functions (subgroups) can also be performed for isospins.

Concluding remarks

In this work,

- we provided an extension of semi-simplicity of spin particle Lie algebra to the Lie group level,
- we showed that a spin particle Lie algebra admits a Clifford algebra, an almost complex manifold and a spin Lie group structure,
- we demonstrated that any spin half particle, (resp. integer spin), spin Lie group is a fourfold, (resp. double), cover of the $\mathrm{SO}(2n)$, (resp. $\mathrm{SO}(2n+1)$), we also proved that any spin Lie group of a spin particle is connected and semi-simple,
- we constructed the real Lie algebra of the Spin $\left(\frac{1}{2}\right)$ particle,
- we also performed the Iwasawa decomposition of the spin half into KDN,
- finally, we applied the angular momentum coupling to the Spin $\left(\frac{2n-1}{2}\right)$ particle and demonstrated that the orthogonal basis transforms into the $\mathrm{SO}(2n)$ one, which is nothing but the Dynkin's root D_n.

THANK YOU FOR YOUR ATTENTION(XIEXIE)