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A Combinatorial Formula for the Associated Legendre Functions of Integer Degree

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We apply inverse scattering theory to a Schrödinger operator with a regular reflectionless Pöschl–Teller potential on the line, to arrive at a combinatorial formula for the associated Legendre functions of integer degree. The expansion coefficients in the combinatorial formula are identified as dimensions of irreducible representations of $gl(N)$, where N corresponds to the degree of the associated Legendre function. As an application, combinatorial formulas for the zonal spherical functions on the real hyperboloids $H^{2N+3,1}=SO_0(2N+3,1;\mathbb{R})/SO_0(2N+2,1;\mathbb{R})$, $H_+^{1,2N+3}=SO_0(2N+3,1;\mathbb{R})/SO(2N+3;\mathbb{R})$ and the sphere $S^{2N+3}=SO(2N+4;\mathbb{R})/SO(2N+3;\mathbb{R})$ are presented. © 2000 Academic Press

1. INTRODUCTION

It is known that the Jost eigenfunctions of the one-dimensional Schrödinger operator with a regular Pöschl–Teller potential can be

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conveniently written in terms of associated Legendre functions (see e.g., [T, Chap. 4.19; F1, Problem 39; AGI; and DJ, Chap. 3.2]). The case of integer-degree associated Legendre functions corresponds in this connection to the situation of a reflectionless Pöschl–Teller potential.

In this paper, we will employ inverse scattering machinery [SCM, DT, AS1, NMPZ] to reconstruct—for the reflectionless case—both the potential and the Jost eigenfunction from the spectral data of the Pöschl–Teller potential. This approach leads us to a combinatorial representation for the integer-degree associated Legendre functions, in which the expansion coefficients turn out to be dimensions of irreducible representations of $gl(N)$. Here the dimension N corresponds to the degree of the associated Legendre function under consideration.

Associated Legendre functions of integer degree N may be used to express the zonal spherical functions [H1, H2, HS] on the real hyperboloids $H^{2N+3,1} = SO_0(2N+3, 1; \mathbb{R})/SO_0(2N+2, 1; \mathbb{R})$, $H_+^{1,2N+3} = SO_0(2N+3, 1; \mathbb{R})/SO(2N+3; \mathbb{R})$ and the sphere $S^{2N+3} = SO(2N+4; \mathbb{R})/SO(2N+3; \mathbb{R})$, respectively. Hence, as a corollary, we also find combinatorial formulas for the zonal spherical functions on these rank-one symmetric spaces.

The paper is organized as follows. Section 2 serves as a reminder of some characteristics of the Schrödinger equation with a regular Pöschl–Teller potential on the line; in particular, its solution in terms of associated Legendre functions is recalled. In Section 3 we have collected some formulas from the inverse scattering theory for one-dimensional Schrödinger operators with reflectionless (Bargmann) potentials; these formulas permit one to reconstruct both the potential (via the Hirota formula [H3]) and the Jost eigenfunction (via the Sato formula [S, SS, DKJM, SW, OSTT]) from the spectral data. A proof of the Sato reconstruction formula for the reflectionless Jost function by means of inverse scattering theory can be found in Appendix A. The derivation given there complements the known inverse-scattering proof of the Hirota reconstruction formula for the corresponding reflectionless Schrödinger potential [SCM, AS1]. Plugging the spectral data of the Pöschl–Teller potential (from Section 2) into the Sato formula for the Jost eigenfunction (from Section 3), produces a combinatorial formula for of the integer-degree associated Legendre functions; some salient properties of this combinatorial formula are analyzed in Section 4. Finally, in Section 5, we apply our combinatorial formula for the associated Legendre functions to arrive at analogous explicit combinatorial representations for the zonal spherical functions on certain hyperboloids and spheres.

2. THE SCHRÖDINGER EQUATION WITH A REGULAR PÖSCHL–TELLER POTENTIAL

The Schrödinger equation with a regular Pöschl–Teller potential on the line is given by (see [T, Chap. 4.19; F1, Problem 39; AGI; and DJ, Chap. 3.2])

$$\left(\frac{d^2}{dx^2}+\frac{N(N+1)}{\cosh^2(x)}-z^2\right)\Psi_N(x,z)=0, \qquad -\infty<x<+\infty. \tag{2.1}$$

Here N denotes a nonnegative coupling parameter determining the strenght of the potential and z is a (generic complex) spectral parameter. To solve Eq. (2.1) it is convenient to perform a change of variables of the form $y=\tanh(x)$. This substitution transforms the Schrödinger equation into the differential equation for the associated Legendre functions of order z and degree N:

$$\Psi_N(x,z)=\phi_N(\tanh(x),z), \tag{2.2a}$$

$$\left((1-y^2)\frac{d^2}{dy^2}-2y\frac{d}{dy}+N(N+1)-\frac{z^2}{(1-y^2)}\right)\phi_N(y,z)=0 \tag{2.2b}$$

(cf. [GR, Eq. 8.700.1]). Two linearly independent solutions of Eq. (2.2b) are given by the associated Legendre functions of the first and second kind, respectively. For our purposes it suffices to consider only the case of an associated Legendre function of the first kind $P_N^z(y)$. In terms of this function the solution to the original Schrödinger equation in (2.1) becomes

$$\begin{aligned}\Psi_N(x,z)&=\Gamma(1-z)\,P_N^z(\tanh(x))\\&=\left(\frac{1+\tanh(x)}{1-\tanh(x)}\right)^{z/2}F\left(-N,N+1;1-z;\frac{1-\tanh(x)}{2}\right)\\&=(e^x+e^{-x})^z\,F\left(N+1-z,-N-z;1-z;\frac{1-\tanh(x)}{2}\right)\\&=\frac{\exp(zx)}{(1+e^{-2x})^N}F(-N,-N-z;1-z;-e^{-2x}),\end{aligned} \tag{2.3}$$

where we have chosen the normalization such that no gamma-function factors appear in front of the hypergeometric-series representations. The formula on the second line of Eq. (2.3) corresponds to the standard hypergeometric representation for the associated Legendre function $P_N^z(\cdot)$ (cf. [GR, Eq. 8.704]); the formulas on the third and fourth lines are obtained from this standard representation by means of linear transformation formulas for the Gauss hypergeometric function. (Specifically, one passes from the second line to the third via the transformation [GR, Eq. 9.131.1 (3rd line)] and from the third line to the fourth via the transformation [GR, Eq. 9.131.1 (2nd line)].)

From now on we shall assume (unless explicitly stated otherwise) that the coupling parameter N is in fact a nonnegative *integer*. In this situation the hypergeometric series on the last line of Eq. (2.3) terminates, and we can write more explicitly

$$\Psi_N(x, z) = \exp(zx) \frac{\sum_{m=0}^{N} C_m^N(z) \binom{N}{m} e^{-2mx}}{\sum_{m=0}^{N} \binom{N}{m} e^{-2mx}}, \tag{2.4a}$$

$$C_m^N(z) = \prod_{j=1}^{m} \left(\frac{z+N+1-j}{z-j} \right) \tag{2.4b}$$

(with the convention that empty products are equal to 1).

From Eqs. (2.4a) and (2.4b) it is immediate that $\Psi_N(x, z)$ has the following plane-wave asymptotics for $x \to \pm\infty$:

$$\Psi_N(x, z) \to \begin{cases} \exp(zx) & \text{for} \quad x \to \infty, \\ \prod_{j=1}^{N} \left(\frac{z+j}{z-j} \right) \exp(zx) & \text{for} \quad x \to -\infty. \end{cases} \tag{2.5}$$

For $z = -j$, $j = 1, ..., N$, the order of the asymptotics for $x \to -\infty$ is lower than the generic value $O(e^{zx})$. These values of the spectral parameter z constitute the discrete spectrum of the Schrödinger operator: the corresponding (bound-state) wave functions are square integrable. (The square integrability of $\Psi_N(x, -j)$, $j = 1, ..., N$, is immediate from Eqs. (2.4a) and (2.4b).) For later use, we will compute the normalization constants of the bound-state wave functions

$$\nu_j = \left(\int_{-\infty}^{\infty} \Psi_N^2(x, -j)\, dx \right)^{-1} = j \binom{2j}{j} \binom{N+j}{N-j}, \qquad j = 1, ..., N. \tag{2.6}$$

Remark 2.1. To infer the normalization formulas in (2.6), it is helpful to note that

$$\begin{aligned} \Psi_N(x, -j) &= j!\, P_N^{-j}(\tanh(x)) \\ &= (-1)^j j! \frac{(N-j)!}{(N+j)!} P_N^j(\tanh(x)) \end{aligned}$$

(cf. [GR, Eq. 8.737.1]) and that $P_N^j(-y)=(-1)^{N+j}\,P_N^j(y)$ (cf. [GR, Eq. 8.737.2]). With these properties of the associated Legendre function in mind, Eq. (2.6) is immediate from the integration formula $\int_0^1 (P_N^j(y))^2 (1-y^2)^{-1}\,dy=(2j)^{-1}(N+j)!/(N-j)!$ (cf. [GR, Eq. 7.122.1]).

Remark 2.2. For a general nonnegative (not necessarily integer-valued) coupling parameter N, the asymptotics of the wave function $\Psi_N(x,z)$ for $x\to\pm\infty$ follows from the fourth line of Eq. (2.3) and the $F(-,-;-;\xi)\to F(-,-;-;1/\xi)$ hypergeometric transformation formula (cf. [GR, Eq. 9.132.2])

$$\Psi_N(x,z)=a(z)\exp(zx)\,\frac{F(-N,\,-N+z;\,1+z;\,-e^{2x})}{(1+e^{2x})^N}$$
$$-b(-z)\exp(-zx)\,\frac{F(-N,\,-N-z;\,1-z;\,-e^{2x})}{(1+e^{2x})^N},$$

with

$$a(z)=\frac{\Gamma(1-z)\,\Gamma(-z)}{\Gamma(1+N-z)\,\Gamma(-N-z)}=\left(1-\frac{\sin^2(\pi N)}{\sin^2(\pi z)}\right)\frac{\Gamma(1+N+z)\,\Gamma(-N+z)}{\Gamma(1+z)\,\Gamma(z)},$$

$$b(z)=-\frac{\Gamma(1+z)\,\Gamma(-z)}{\Gamma(1+N)\,\Gamma(-N)}=-\frac{\sin(\pi N)}{\sin(\pi z)}$$

(where the reflection relation $\Gamma(\xi)\,\Gamma(1-\xi)=\pi/\sin(\pi\xi)$ was used to rewrite the products of gamma functions). We read-off the asymptotics

$$\Psi_N(x,z)\to\begin{cases}\exp(zx) & \text{for}\quad x\to\infty,\\ a(z)\exp(zx)-b(-z)\exp(-zx) & \text{for}\quad x\to-\infty.\end{cases}\tag{2.7}$$

For N integer we have that $b(z)=0$, i.e., the potential is reflectionless and the asymptotics in (2.7) reduces to that of (2.5).

3. THE HIROTA AND SATO FORMULAS

The qualitative behavior of the solutions of the Schrödinger equation with a regular Pöschl–Teller potential is characteristic for that of more general Schrödinger equations with rapidly decaying potentials (see e.g., [SCM, DT, AS1, NMPZ]). In particular, for a potential $u\in\mathcal{S}(\mathbb{R})$ (i.e., the potential $u(x)$ is assumed to be real, smooth, and integrable against polynomials in x) the solutions of the Schrödinger equation

$$\left(\frac{d^2}{dx^2}+u(x)-z^2\right)\Psi(x,z)=0, \qquad -\infty<x<+\infty \tag{3.1}$$

are asymptotically free: $\Psi(x,z)\to\alpha_\pm(z)\,e^{zx}+\beta_\pm(z)\,e^{-zx}$ for $x\to\pm\infty$. The solution of (3.1) with plane-wave asymptotics of the form $\exp(zx)$, $x\to\infty$, is called the Jost solution (of the first kind). One has

$$\Psi_{jost}(x,z)\to\begin{cases}\exp(zx) & \text{for} \quad x\to\infty,\\ a(z)\exp(zx)-b(-z)\exp(-zx) & \text{for} \quad x\to-\infty.\end{cases} \tag{3.2}$$

The quotient $r(z)=b(z)/a(z)$ is referred to as the reflection coefficient; when $b(z)=r(z)=0$ the potential is said to be reflectionless. The zeros of $a(z)$ on the negative real axis correspond to the discrete spectrum of the Schrödinger operator; for these values of the spectral parameter z the Jost function is square integrable. For potentials $u\in\mathcal{S}(\mathbb{R})$, there are at most a finite number of such square integrable eigenfunctions.

Let $-\kappa_1, ..., -\kappa_N$ be the zeros of $a(z)$ and let $\nu_1, ..., \nu_N$ denote the corresponding normalization constants

$$\nu_j=\left(\int_{-\infty}^{\infty}\Psi_{jost}^2(x,-\kappa_j)\,dx\right)^{-1}, \qquad j=1, ..., N. \tag{3.3}$$

The numbers κ_j, ν_j and the reflection coefficient $r(z)=b(z)/a(z)$ are referred to as the spectral data of the potential. A deep fundamental result from the inverse scattering theory for one-dimensional Schrödinger equations states that, for a Schwartz class potential $u\in\mathcal{S}(\mathbb{R})$, both the potential $u(x)$ and the Jost solution $\Psi_{jost}(x,z)$ can be completely recovered from the spectral data (see e.g., [SCM, DT, AS1, NMPZ]).

For reflectionless (or Bargmann) potentials the reconstruction of the potential and Jost function from the discrete spectrum and the normalization constants is in fact completely explicit. Specifically, the Bargmann potential is given by the *Hirota Formula*

$$u(x)=2\frac{d^2}{dx^2}\log\tau(x) \tag{3.4a}$$

$$\tau(x)=\sum_{J\subset\{1,...,N\}}\prod_{j\in J}\frac{\nu_j}{2\kappa_j}\prod_{\substack{j,k\in J\\ j<k}}\left(\frac{\kappa_j-\kappa_k}{\kappa_j+\kappa_k}\right)^2\exp\left(-2\sum_{j\in J}\kappa_j x\right) \tag{3.4b}$$

and the corresponding Jost function is given by the *Sato Formula*

$$\Psi_{jost}(x, z) = \exp(zx) \times \frac{\displaystyle\sum_{J \subset \{1, ..., N\}} \prod_{j \in J} \frac{\nu_j}{2\kappa_j} \left(\frac{z+\kappa_j}{z-\kappa_j}\right) \prod_{\substack{j, k \in J \\ j<k}} \left(\frac{\kappa_j - \kappa_k}{\kappa_j + \kappa_k}\right)^2 \exp\left(-2 \sum_{j \in J} \kappa_j x\right)}{\displaystyle\sum_{J \subset \{1, ..., N\}} \prod_{j \in J} \frac{\nu_j}{2\kappa_j} \prod_{\substack{j, k \in J \\ j<k}} \left(\frac{\kappa_j - \kappa_k}{\kappa_j + \kappa_k}\right)^2 \exp\left(-2 \sum_{j \in J} \kappa_j x\right)}. \tag{3.5}$$

The derivation of the Hirota formula for the reflectionless potential by means of inverse scattering theory is quite standard [SCM, DT, AS1, NMPZ]. The derivation of the Sato formula for the corresponding Jost function by means of inverse scattering methods, however, does not appear to be standard. We have therefore included this derivation in Appendix A.

Remark 3.1. The Hirota and Sato formulas express the fact that a Schrödinger equation (3.1) with potential of the Hirota form (3.4a), (3.4b) has a Jost solution given by the Sato formula (3.5). The proof by inverse scattering (cf. Appendix A) initially covers only the case of positive spectral data κ_j, ν_j. However, it is clear that the Schrödinger equation (3.1) with a Hirota potential and Sato wave function then in fact holds as a rational identity in κ_j, ν_j and $\exp(-2\kappa_j x)$.

Remark 3.2. Both the Hirota formula and the Sato formula have their origin in the theory of nonlinear integrable wave equations of KdV type [H3, S, SS, DKJM, SW, OSTT]. The usual proof that the Sato wave function solves the Schrödinger equation with Hirota potential uses geometric and/or representation-theoretic methods [S, SS, DKJM, SW]. To make the contact with this previous literature more transparent, it is useful to introduce the (KdV N-soliton) tau function

$$\tau(\theta_1, ..., \theta_N) = \sum_{J \subset \{1, ..., N\}} \prod_{\substack{j, k \in J \\ j<k}} \left(\frac{\kappa_j - \kappa_k}{\kappa_j + \kappa_k}\right)^2 \prod_{j \in J} n_j e^{-2\theta_j}. \tag{3.6}$$

The Sato theory, developed (mainly) by the Kyoto school [S, SS, DKJM, SW], then states that the Schrödinger equation (3.1) with a (KdV N-soliton) potential of the Hirota form

$$u_{hirota}(x) = 2 \frac{d^2}{dx^2} \log \tau(\kappa_1 x, ..., \kappa_N x) \tag{3.7}$$

is solved by the Sato wave function

$$\Psi_{sato}(x,z)=\exp(zx)\times\frac{\tau\left(\kappa_1 x-\sum_{m=1,\, m\ odd}^{\infty}\frac{1}{m}\left(\frac{\kappa_1}{z}\right)^m, \ldots, \kappa_N x-\sum_{m=1,\, m\ odd}^{\infty}\frac{1}{m}\left(\frac{\kappa_N}{z}\right)^m\right)}{\tau(\kappa_1 x, \ldots, \kappa_N x)}. \tag{3.8}$$

After the parametrization $n_j=\nu_j/(2\kappa_j)$, the Hirota potential in (3.7) goes over in the potential of (3.4a), (3.4b), and the Sato wave function in (3.8) passes over to the wave function of (3.5) in view of the fact that

$$\sum_{m=1,\, m\ odd}^{\infty}\frac{1}{m}\left(\frac{\kappa_j}{z}\right)^m=\frac{1}{2}\log\left(\frac{z+\kappa_j}{z-\kappa_j}\right)\qquad(|z|>\kappa_j). \tag{3.9}$$

Remark 3.3. The inverse-scattering approach towards the Sato formula, adopted in Appendix A, yields a parametrization of the reflectionless Sato Jost function in terms of the spectral data (i.e., the eigenvalues and normalization constants). It is indeed clear from the explicit formula in (3.5) that the spectral values $z=-\kappa_j$, $j=1, \ldots, N$ correspond to the discrete spectrum of the Schrödinger operator, as at these values for z the wave function becomes exponentially decaying for $x\to\pm\infty$ (and hence square-integrable). The parameters ν_j, $j=1, \ldots, N$ correspond to the associated normalization constants (cf. Eq. 3.3). We therefore conclude that the following nonobvious integration formulas hold for the reflectionless Jost function $\Psi_{jost}(x,z)$ given by the Sato formula in (3.5):

$$\int_{-\infty}^{\infty}\Psi_{jost}^2(x,-\kappa_j)\,dx=1/\nu_j,\qquad j=1, \ldots, N \tag{3.10}$$

(i.e., these integration formulas *follow* from the explicit reconstruction of the Jost function $\Psi_{jost}(x,z)$ from the spectral data, given in Appendix A).

4. A COMBINATORIAL REPRESENTATION OF THE INTEGER-DEGREE ASSOCIATED LEGENDRE FUNCTION

If we plug the spectral data of the Pöschl–Teller potential into the Sato formula for the reflectionless Jost function, then we end up with a combinatorial expression for the (renormalized) associated Legendre functions of integer degree.

PROPOSITION 4.1. *Let* $\Psi_N(x, z) = \Gamma(1-z)\, P_N^z(\tanh(x))$, *where* $P_N^z(y)$ *denotes the associated Legendre function of degree* $N \in \mathbb{N}^*$ *and order* $z \in \mathbb{C} \setminus \mathbb{N}^*$. *Then we have that*

$$\Psi_N(x, z) = \exp(zx) \times \frac{\sum_{J \subset \{1, ..., N\}} \prod_{j \in J} \left(\frac{z+j}{z-j}\right) \prod_{j \in J,\, k \notin J} \left|\frac{j+k}{j-k}\right| \exp\left(-2x \sum_{j \in J} j\right)}{\sum_{J \subset \{1, ..., N\}} \prod_{j \in J,\, k \notin J} \left|\frac{j+k}{j-k}\right| \exp\left(-2x \sum_{j \in J} j\right)}$$

(*where*, *by convention*, *empty products are equal to* 1).

Proof. For the Schrödinger equation with a regular Pöschl–Teller potential, the values of the spectral parameter corresponding to the discrete spectrum are given by $z = -\kappa_j$, $j = 1, ..., N$, with $\kappa_j = j$ (see Section 2.). The values of the associated normalization constants read (cf. Eq. 2.6)

$$\nu_j = j \binom{2j}{j} \binom{N+j}{N-j} = 2j \prod_{\substack{1 \leqslant k \leqslant N \\ k \neq j}} \left|\frac{j+k}{j-k}\right|.$$

Substitution of these spectral data into the Sato formula (3.5), produces a solution to the Pöschl–Teller Schrödinger equation (2.1) of the form stated by the proposition. The solution in question has the same Jost-function asymptotics as the previously given solution of Section 2. in terms of associated Legendre functions (cf. Eqs. (2.3) and (2.5)). Hence, both solutions coincide. ∎

It turns out that the expansion coefficients of the form $\prod_{j \in J, k \notin J} |(j+k)/(j-k)|$ appearing in the combinatorial formula of Proposition 4.1 are in fact integers. The following proposition identifies these numbers as dimensions of irreducible representations of the Lie algebra $gl(N)$. The weights/partitions of the representations at issue are easiest characterized in terms of their Frobenius symbol (see e.g., [M, Chap. I.1] and Remark 4.2 below).

PROPOSITION 4.2. *Let* $J = \{1 \leqslant j_1 < j_2 < \cdots < j_r \leqslant N\}$ *and let* λ_J *denote the partition with Frobenius symbol* $(j_r, j_{r-1}, ..., j_1 \mid j_r - 1, j_{r-1} - 1, ..., j_1 - 1)$. *Then the dimension of the irreducible representation of the Lie algebra* $gl(N)$ *corresponding to the highest weight* λ_J *is given by*

$$\dim V_{\lambda_J}^{gl(N)} = \prod_{\substack{j \in J \\ k \in \{1, ..., N\} \setminus J}} \left|\frac{j+k}{j-k}\right|.$$

Proof. The dimension of the irreducible representation of $gl(N)$ associated to a partition λ is given by the hook formula (see e.g., [F2, Chap. 4.3])

$$\dim V_\lambda^{gl(N)} = \prod_{(i,\,j)\in\lambda} \left(\frac{N+c(i,\,j)}{h(i,\,j)}\right), \tag{4.1}$$

where $c(i, j)=j-i$ and $h(i, j)=\lambda_i+\lambda'_j-i-j+1$. (Here the product in the hook formula is meant over all boxes (i, j) in the Young diagram of λ and the partition λ' denotes the conjugate of λ, i.e., the partition corresponding to the transpose of the Young diagram of λ.)

Let us write $d_J(N)=\dim V_{\lambda_J}^{gl(N)}$ and $n_J(N)=\prod_{j\in J,\,k\notin J}|(j+k)/(j-k)|$. We will now use the hook formula to demonstrate that

$$\frac{d_J(N)}{d_{J'}(N)}=\frac{n_J(N)}{n_{J'}(N)}, \tag{4.2}$$

where $J'=J\backslash\{j_r\}=\{j_1, ..., j_{r-1}\}$. From this relation the statement of the proposition is immediate by induction on the cardinality r of the index set J, starting from the initial values for $r=0$ given by $d_\varnothing(N)=n_\varnothing(N)=1$.

To infer the relation in (4.2), it is useful to observe that the partitions λ_J and $\lambda_{J'}$ differ by the hook $(j_r \mid j_r-1)$ formed by the first row and column of λ_J. Consequently, the hook formula in (4.1) entails the following expression for the l.h.s. of (4.2)

$$\begin{aligned}\frac{d_J(N)}{d_{J'}(N)}&=\frac{N+c(1,1)}{h(1,1)}\prod_{1<i\leqslant j_r}\left(\frac{N+c(i,1)}{h(i,1)}\right)\prod_{1<j\leqslant j_r+1}\left(\frac{N+c(1,j)}{h(1,j)}\right)\\&=\frac{1}{2j_r}\frac{(N+j_r)!}{(N-j_r)!}\prod_{1<i\leqslant j_r}h^{-1}(i,1)\prod_{1<j\leqslant j_r+1}h^{-1}(1,j)\\&=\frac{1}{2(j_r!)^2}\frac{(N+j_r)!}{(N-j_r)!}\prod_{1\leqslant s<r}\left(\frac{j_r-j_s}{j_r+j_s}\right)^2,\end{aligned}$$

where we have used that for the partition λ_J

$$\prod_{1<j\leqslant j_r+1}h(1,j)=j_r\prod_{1<i\leqslant j_r}h(i,1)=j_r!\prod_{1\leqslant s<r}\left(\frac{j_r+j_s}{j_r-j_s}\right)$$

(cf. Eq. 4.4 below). Moreover, the r.h.s. of (4.2) readily produces

$$\begin{aligned}\frac{n_J(N)}{n_{J'}(N)}&=\prod_{k\notin J}\left|\frac{j_r+k}{j_r-k}\right|\prod_{j\in J'}\left|\frac{j-j_r}{j+j_r}\right|=\prod_{\substack{1\leqslant k\leqslant N\\ k\neq j_r}}\left|\frac{j_r+k}{j_r-k}\right|\prod_{j\in J'}\left(\frac{j_r-j}{j_r+j}\right)^2\\&=\frac{1}{2(j_r!)^2}\frac{(N+j_r)!}{(N-j_r)!}\prod_{1\leqslant s<r}\left(\frac{j_r-j_s}{j_r+j_s}\right)^2.\end{aligned}$$

(In the above formulas it is understood that the for $r=1$ empty product of the form $\prod_{1\leqslant s<r}(j_r-j_s)^2/(j_r+j_s)^2$ is interpreted as 1.) Hence, we have (4.2) and thus (by induction on r) $d_J(N)=n_J(N)$. ∎

The interpretation of the expansions coefficients as dimensions of irreducible representations of $gl(N)$ permits one to recast the combinatorial formula of Proposition 4.1 in terms of Schur functions. Let us to this end recall that the Schur function $s_\lambda(\mathbf{x})=s_\lambda(x_1, ..., x_N)$, associated to a partition $\lambda=(\lambda_1, ..., \lambda_N)$, is given explicitly by [M, Chap. I.3]

$$s_\lambda(\mathbf{x})=\frac{\det[x_k^{\rho_j+\lambda_j}]_{1\leqslant j,k\leqslant N}}{\det[x_k^{\rho_j}]_{1\leqslant j,k\leqslant N}} \tag{4.3a}$$

$$=\frac{\sum_{\sigma\in S_N}(-1)^\sigma x_{\sigma_1}^{\rho_1+\lambda_1}\cdots x_{\sigma_N}^{\rho_N+\lambda_N}}{\sum_{\sigma\in S_N}(-1)^\sigma x_{\sigma_1}^{\rho_1}\cdots x_{\sigma_N}^{\rho_N}}, \tag{4.3b}$$

with $\rho=(N-1, N-2, ..., 1, 0)$.

Corollary 4.3. *The renormalized associated Legendre function* $\Psi_N(x,z)=\Gamma(1-z)\,P_N^z(\tanh(x))$, *with* $N\in\mathbb{N}^*$ *and* $z\in\mathbb{C}\backslash\mathbb{N}^*$, *can be written compactly in terms of Schur functions as follows*

$$\Psi_N(x,z)=\exp(zx)\,\frac{\sum_\lambda c_\lambda(z)\,s_\lambda(e^{-x}\mathbf{1}_N)}{\sum_\lambda s_\lambda(e^{-x}\mathbf{1}_N)}.$$

Here the range of the summations in numerator and denominator is over all partitions $\lambda=(\lambda_1, ..., \lambda_N)$ *with a Frobenius symbol of the form* $(\alpha_1, ..., \alpha_r \mid \alpha_1-1, ..., \alpha_r-1)$ *with* $\alpha_1\leqslant N$ *(and* $r=0, ..., N$*). Furthermore, the coefficient function* $c_\lambda(z)$ *is defined by*

$$c_\lambda(z)=\left(\frac{z+\alpha_1}{z-\alpha_1}\right)\cdots\left(\frac{z+\alpha_r}{z-\alpha_r}\right)$$

and the vector $\mathbf{1}_N$ *represents the N-dimensional vector with unit components.*

Proof. The Weyl character formula for $GL(N)$ (see e.g., [FH]) produces, when specialized to the identity element, the well-known (Weyl-) dimension formula $\dim(V_\lambda^{gl(N)})=s_\lambda(\mathbf{1}_N)$. From this, and the expression for the dimension in Proposition 4.2, it follows that the formula stated in the corollary reduces to that of Proposition 4.1 upon employing the fact that the Schur function $s_\lambda(\mathbf{x})$, associated to a partition λ with Frobenius symbol $(\alpha_1, ..., \alpha_r \mid \alpha_1 - 1, ..., \alpha_r - 1)$, is homogeneous of degree $|\lambda| = \lambda_1 + \cdots + \lambda_N = 2(\alpha_1 + \cdots + \alpha_r)$. ∎

Remark 4.1. In Proposition 4.1 and Corollary 4.3 the combinatorial formula is—for reasons of convenience—formulated only for *positive* values of the degree N. However, the stated formulas in fact also cover the trivial case $N = 0$ (this corresponds to the free Schrödinger equation with potential $u(x) = 0$). Indeed, for $N = 0$ the combinatorial formula reduces to the well-known relation $\Gamma(1-z)\, P_0^z(\tanh(x)) = \exp(zx)$ (cf. [GR, Eq. 8.753.1]).

Remark 4.2. In Proposition 4.2 and Corollary 4.3 we used the notation of Frobenius symbols [M, Ch. I.1] to characterize our partitions. Let $\lambda = (\lambda_1, ..., \lambda_N)$ be a partition (so the components λ_j are integers such that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_N \geqslant 0$) and let us assume that the diagonal of the corresponding Young diagram consist of say r boxes (so $0 \leqslant r \leqslant N$). Then λ is completely characterized by the numbers $\alpha_j = \lambda_j - j$ (counting the number of boxes in the jth row to the right of the diagonal box (j, j)) and $\beta_j = \mathrm{Card}\{k \mid \lambda_k \geqslant j\}$ (counting the number of boxes in the jth column below the diagonal box (j, j)), with $j = 1, ..., r$. Reversely, to every pair of r ($\leqslant N$) strictly decreasing nonnegative integers $\alpha_1 > \cdots > \alpha_r \geqslant 0$ and $N > \beta_1 > \cdots > \beta_r \geqslant 0$ there corresponds a unique partition $\lambda = (\lambda_1, ..., \lambda_N)$. One writes $\lambda = (\alpha_1, ..., \alpha_r \mid \beta_1, ..., \beta_r)$, which is referred to as the Frobenius symbol of λ.

For example, a partition $\lambda = (\lambda_1, ..., \lambda_N)$ with Frobenius symbol of the form $(\alpha_1, ..., \alpha_r \mid \alpha_1 - 1, ..., \alpha_r - 1)$ corresponds to the partition

$$\lambda = (\alpha_1 + 1, \alpha_2 + 2, ..., \alpha_r + r, \underbrace{r, ..., r}_{\alpha_r - 1}, \underbrace{r-1, ..., r-1}_{\alpha_{r-1} - \alpha_r - 1}, ..., \underbrace{1, ..., 1}_{\alpha_1 - \alpha_2 - 1}, \underbrace{0, ..., 0}_{N - \alpha_1}) \tag{4.4}$$

(cf. Fig. 1).

Remark 4.3. If we substitute the spectral data of the reflectionless Pöschl–Teller potential $N(N+1)/\cosh^2(x)$ in Eqs. (3.4a) and (3.4b), then

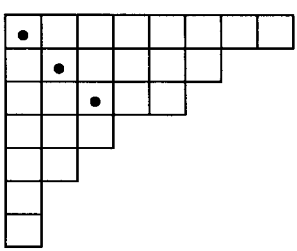
$(7,4,2 \mid 6,3,1) =$  $= (\underbrace{8,6,5,3,2,1,1,0,\ldots,0}_{N})$

FIG. 1. The Frobenius symbol and Young diagram of the partition (8, 6, 5, 3, 2, 1, 1, 0, ..., 0).

we arrive at a representation of Hirota form for the potential under consideration. In other words, this produces an expression for the Pöschl–Teller potential in terms of the second-order logarithmic derivative of the tau function. Integrating twice leads us to the identity

$$(1+e^{-2x})^{N(N+1)/2} = \sum_{J\subset\{1,\dots,N\}} \prod_{j\in J,\, k\notin J} \left|\frac{j+k}{j-k}\right| \exp\left(-2x\sum_{j\in J} j\right)$$

$$=\sum_{\lambda} s_\lambda(e^{-x}\mathbf{1}_N), \tag{4.5}$$

where on the second line the summation is over all partitions $\lambda=(\lambda_1, \dots, \lambda_N)$ with a Frobenius symbol of the form $(\alpha_1, \dots, \alpha_r \mid \alpha_1-1, \dots, \alpha_r-1)$ with $\alpha_1 \leqslant N$ and $r=0, \dots, N$. (The integration constants are determined by the asymptotics for $x\to\infty$ and the formula on the second line follows from that on the first line by the same reasoning as in the proof of Corollary 4.3.)

This formula may be viewed as the C_N-type Weyl denominator formula in the appearance [M, p. 79]

$$\prod_{1\leqslant j\leqslant N} (1+y_j^2) \prod_{1\leqslant j<k\leqslant N} (1+y_j y_k) = \sum_{\lambda} s_\lambda(y_1, \dots, y_N) \tag{4.6}$$

specialized to the diagonal $\mathbf{y}=e^{-x}\mathbf{1}_N$ (where the range of the summation on the r.h.s. of Eq. (4.6) is the same as on the second line of Eq. (4.5)).

Remark 4.4. If the denominator of the hypergeometric representation (2.4a), (2.4b) for the renormalized associated Legendre function $\Psi_N(x,z)=\Gamma(1-z)\, P_N^z(\tanh(x))$, viz. $(1+e^{-2x})^N$, is compared with the denominator of the combinatorial representation of Proposition 4.1, viz. $(1+e^{-2x})^{N(N+1)/2}$ (cf. Eq. (4.5)), then one observes that both denominators differ by a factor $(1+e^{-2x})^{N(N-1)/2}$. Thus, since both formulas represent the function $\Psi_N(x,z)$, the corresponding numerators must also differ by that same factor and we have

$$\sum_{J\subset\{1,\dots,N\}} \prod_{j\in J} \left(\frac{z+j}{z-j}\right) \prod_{j\in J,\, k\notin J} \left|\frac{j+k}{j-k}\right| \exp\left(-2x\sum_{j\in J} j\right)$$

$$=(1+\exp(-2x))^{N(N-1)/2} \sum_{m=0}^{N} \prod_{j=1}^{m} \left(\frac{z+N+1-j}{z-j}\right)\binom{N}{m} \exp(-2mx). \tag{4.7}$$

In other words, one descends from the combinatorial representation of Proposition 4.1 to the more conventional hypergeometric representation

(2.4a), (2.4b) by dividing out a common factor $(1+e^{-2x})^{N(N-1)/2}$ in the numerator and denominator. The price paid for such condensation of the formula for the associated Legendre function is, of course, that the combinatorial structure is lost.

Remark 4.5. It is well-known that the dimensions of irreducible representations of $gl(N)$ admit a combinatorial interpretation in terms of the number of semi-standard Young tableaux of given shape (see e.g. [F2, Chap. 4.3]). Hence, as a consequence of Proposition 4.2, the expansion coefficients appearing in the combinatorial formula of Proposition 4.1 also admit such combinatorial interpretation. Indeed, one concludes from Proposition 4.2 that the integer $\prod_{j\in J,\, k\notin J} |(j+k)/(j-k)|$ associated to a given index set $J=\{1\leqslant j_1<\cdots<j_r\leqslant N\}$ amounts to the number of semi-standard Young tableaux of shape $\lambda_J=(j_r, ..., j_1 \mid j_r-1, ..., j_1-1)$ with entries not exceeding N. In other words, the expansion coefficient counts the number of ways in which the boxes of the Young diagram corresponding to the partition λ_J can be assigned numbers from 1 to N, in such a way that the numbers strictly increase along a column (from top to bottom) and do not decrease along a row (from left to right).

Remark 4.6. It turns out possible to generalize the dimension formula of Proposition 4.2 somewhat. Let us to this end denote by $\lambda_J^{(n)}$ the partition in N parts with a Frobenius symbol of the form $(j_r+n, ..., j_1+n \mid j_r-1, ..., j_1-1)$, where n is a nonnegative integer and $J=\{1\leqslant j_1<\cdots<j_r\leqslant N\}$. We then have that the irreducible representation of $gl(N)$ associated to $\lambda_J^{(n)}$ has a dimension given by

$$\dim V^{gl(N)}_{\lambda_J^{(n)}} = \prod_{\substack{j\in J \\ k\in\{1, ..., N\}\setminus J}} \left|\frac{n+j+k}{j-k}\right|. \tag{4.8}$$

For $n=0$ this reduces to the statement of Proposition 4.2. The proof of the dimension formula formulated there also applies (with obvious minor modifications) to the extension in (4.8). One concludes in particular that the r.h.s. of (4.8) is integer-valued for arbitrary $n\in\mathbb{N}$.

5. COMBINATORIAL FORMULAS FOR ZONAL SPHERICAL FUNCTIONS ON REAL ODD-DIMENSIONAL HYPERBOLOIDS AND SPHERES.

In this section we apply the formula of Proposition 4.1 to arrive at combinatorial expressions for the zonal spherical functions on I: the noncompact pseudo-Riemannian symmetric space $SO_0(2N+3, 1; \mathbb{R})/SO_0(2N+2, 1; \mathbb{R})$, II: the noncompact Riemannian symmetric space $SO_0(2N+3, 1; \mathbb{R})/SO(2N+3; \mathbb{R})$, and III: the compact Riemannian symmetric space $SO(2N+4; \mathbb{R})/SO(2N+3; \mathbb{R})$. These symmetric spaces may be realized explicitly as *I*: the one-sheeted hyperboloid $H^{2N+3,1} = \{\mathbf{x} \in \mathbb{R}^{2N+4} \mid x_1^2 + \cdots + x_{2N+3}^2 - x_{2N+4}^2 = 1\}$, II: the upper half (say) $H_+^{1,2N+3} = \{\mathbf{x} \in \mathbb{R}^{2N+4} \mid x_1^2 - x_2^2 - \cdots - x_{2N+4}^2 = 1, x_1 > 0\}$ of the two-sheeted hyperboloid $H^{1,2N+3} = \{\mathbf{x} \in \mathbb{R}^{2N+4} \mid x_1^2 - x_2^2 - \cdots - x_{2N+4}^2 = 1\}$, and III: the sphere $S^{2N+3} = \{\mathbf{x} \in \mathbb{R}^{2N+4} \mid x_1^2 + \cdots + x_{2N+4}^2 = 1\}$, respectively.

The zonal spherical functions are eigenfunctions of the radial part of the Laplace-Beltrami operator on the symmetric space. For the three cases under consideration, this gives rise to the following differential equations for the zonal spherical functions ϕ_N [H1, H2, HS].

Case I. $H^{2N+3,1} = SO_0(2N+3, 1; \mathbb{R})/SO_0(2N+2, 1; \mathbb{R})$

$$\left(\frac{d^2}{dx^2} + 2(N+1)\tanh(x)\frac{d}{dx} + (N+1)^2 - z^2\right)\phi_N^{(1)}(x, z) = 0, \quad -\infty < x < +\infty, \tag{5.1}$$

Case II. $H_+^{1,2N+3} = SO_0(2N+3, 1; \mathbb{R})/SO(2N+3; \mathbb{R})$

$$\left(\frac{d^2}{dx^2} + 2(N+1)\coth(x)\frac{d}{dx} + (N+1)^2 - z^2\right)\phi_N^{(2)}(x, z) = 0, \quad 0 < x < +\infty, \tag{5.2}$$

Case III. $S^{2N+3} = SO(2N+4; \mathbb{R})/SO(2N+3; \mathbb{R})$

$$\left(\frac{d^2}{dx^2} + 2(N+1)\cot(x)\frac{d}{dx} + n(n+2N+2)\right)\phi_N^{(3)}(x, n) = 0, \quad 0 < x < \pi. \tag{5.3}$$

Here z is a complex spectral parameter and n denotes a discrete integer-valued spectral parameter. (For the noncompact pseudo-Riemannian symmetric space of Case I the Laplace–Beltrami operator has both continuous and discrete spectrum, whereas for the Riemannian symmetric spaces of Case II (noncompact) and Case III (compact) the spectrum is completely continuous in the noncompact and completely discrete in the compact case.) By a variable substitution of the form $y = -\sinh^2(x)$ (Cases I and II) or $y = \sin^2(x)$ (Case III), the second-order differential equations (5.1)–(5.3) transform to Gauss hypergeometric equations in y. This leads one to the following hypergeometric representations for the (unnormalized) zonal spherical functions [H1, H2, HS]

Case I. $H^{2N+3,1} = SO_0(2N+3, 1; \mathbb{R})/SO_0(2N+2, 1; \mathbb{R})$

$$\phi^{(1)}_{N,\ even}(x, z) = F\left(\frac{N+1+z}{2}, \frac{N+1-z}{2}; \frac{1}{2}; -\sinh^2(x)\right) \tag{5.4a}$$

$$\phi^{(1)}_{N,\ odd}(x, z) = \sinh(x)\, F\left(1+\frac{N+z}{2}, 1+\frac{N-z}{2}; \frac{3}{2}; -\sinh^2(x)\right), \tag{5.4b}$$

Case II. $H^{1,2N+3}_{+} = SO_0(2N+3, 1; \mathbb{R})/SO(2N+3; \mathbb{R})$

$$\phi^{(2)}_{N}(x, z) = F\left(\frac{N+1+z}{2}, \frac{N+1-z}{2}; N+\frac{3}{2}; -\sinh^2(x)\right), \tag{5.5}$$

Case III. $S^{2N+3} = SO(2N+4; \mathbb{R})/SO(2N+3; \mathbb{R})$

$$\begin{aligned}\phi^{(3)}_{N}(x, n) &= F\left(-\frac{n}{2}, \frac{n}{2}+N+1; N+\frac{3}{2}; \sin(x)\right)\\ &= F\left(-n, n+2(N+1); N+\frac{3}{2}; \sin^2\left(\frac{x}{2}\right)\right).\end{aligned} \tag{5.6}$$

(The differential equation for the pseudo-Riemannian zonal spherical function of Case I (cf. Eq. (5.1)) is regular on the real line $-\infty < x < \infty$. Consequently, one has two independent regular solutions to the eigenvalue equation of the Laplace–Beltrami operator. For the Riemannian Cases II and III, on the other hand, $x = 0$ is a (regular) singular point of the differential equation (cf. Eqs. (5.2), (5.3)). Hence, in these cases one only has one regular solution. In the compact Case III, there is furthermore a (regular) singularity at $x = \pi$. Requiring regularity of the solution also at this second singular point gives rise to the discretization of the spectral variable.) The following three propositions provide a combinatorial representation for the zonal spherical functions (5.4a), (5.4b) (Case I), (5.5) (Case II, and (5.6) (Case III).

PROPOSITION 5.1 (Case I. $H^{2N+3,1} = SO_0(2N+3, 1; \mathbb{R})/SO_0(2N+2, 1; \mathbb{R})$) *Let $N \in \mathbb{N}^*$ and let $z \in \mathbb{C}\backslash\mathbb{N}$. Then one has that*

$$\begin{aligned}&\frac{2^z \Gamma\left(\frac{1+z+N}{2}\right)\Gamma\left(\frac{z-N}{2}\right)}{2^{N+1}\Gamma\left(\frac{1}{2}\right)\Gamma(z)} F\left(\frac{N+1+z}{2}, \frac{N+1-z}{2}; \frac{1}{2}; -\sinh^2(x)\right)\\ &= (\Phi^{(1)}_N(x, z) + \Phi^{(1)}_N(-x, z))\end{aligned}$$

and

$$\frac{2^z\Gamma\left(1+\frac{z+N}{2}\right)\Gamma\left(\frac{1+z-N}{2}\right)}{2^{N+1}\Gamma\left(\frac{3}{2}\right)\Gamma(z)}\sinh(x)\,F\left(1+\frac{N+z}{2},\,1+\frac{N-z}{2};\,\frac{3}{2};\,-\sinh^2(x)\right)$$

$$=(\Phi_N^{(1)}(x,z)-\Phi_N^{(1)}(-x,z)),$$

where

$$\Phi_N^{(1)}(x,z)=\exp((z-N-1)\,x)$$

$$\times\frac{\sum_{J\subset\{1,\ldots,N\}}\prod_{j\in J}\left(\frac{z+j}{z-j}\right)\prod_{j\in J,\,k\notin J}\left|\frac{j+k}{j-k}\right|\exp\left(-2x\sum_{j\in J}j\right)}{\sum_{J\subset\{1,\ldots,N+1\}}\prod_{j\in J,\,k\notin J}\left|\frac{j+k}{j-k}\right|\exp\left(-2x\sum_{j\in J}j\right)}$$

$$=\exp((z-N-1)\,x)\,\frac{\sum_\lambda c_\lambda(z)\,s_\lambda(e^{-x}\mathbf{1}_N)}{\sum_\lambda s_\lambda(e^{-x}\mathbf{1}_{N+1})}.$$

Here, in the expression for $\Phi_N^{(1)}(x,z)$ *in terms of Schur functions* $s_\lambda(\mathbf{x})$ *written on the last line, the notation is in correspondence with that of Corollary* 4.11. *The summation in the numerator is over all partitions* $\lambda=(\lambda_1,\ldots,\lambda_N)$ *with Frobenius symbol of the form* $(\alpha_1,\ldots,\alpha_r\,|\,\alpha_1-1,\ldots,\alpha_r-1)$ $(0\leqslant r\leqslant N)$ *and parts* λ_j *not exceeding* $N+1$ *(i.e.,* $\alpha_1\leqslant N$*). The summation in the denominator is over all partitions* $\lambda=(\lambda_1,\ldots,\lambda_{N+1})$ *with a Frobenius symbol of the form* $(\alpha_1,\ldots,\alpha_r\,|\,\alpha_1-1,\ldots,\alpha_r-1)$ $(0\leqslant r\leqslant N+1)$ *and parts* λ_j *not exceeding* $N+2$ *(i.e.,* $\alpha_1\leqslant N+1$*). Furthermore, the coefficient function* $c_\lambda(z)$ *is given by*

$$c_\lambda(z)=\left(\frac{z+\alpha_1}{z-\alpha_1}\right)\cdots\left(\frac{z+\alpha_r}{z-\alpha_r}\right),$$

and the vectors $\mathbf{1}_N$ *and* $\mathbf{1}_{N+1}$ *represent* N*- and* $N+1$*-dimensional vectors with all components equal to* 1*, respectively.*

Proof. One has that

$$\Psi_N(x,z)=\Gamma(1-z)\,P_N^z(\tanh(x))$$

$$\overset{\text{(i)}}{=}(e^x+e^{-x})^z\,F\left(N+1-z,\,-N-z;\,1-z;\,\frac{1-\tanh(x)}{2}\right)$$

$$\overset{\text{(ii)}}{=} (e^x - e^{-x})^z \coth^{N+1}(x)\, F\left(\frac{N+1+z}{2}, \frac{N+1-z}{2}; 1-z; -\frac{1}{\sinh^2(x)}\right)$$

$$\overset{\text{(iii)}}{=} c_{even} \cosh^{N+1}(x)\, F\left(\frac{N+1+z}{2}, \frac{N+1-z}{2}; \frac{1}{2}; -\sinh^2(x)\right)$$

$$+ c_{odd} \cosh^{N+1}(x) \sinh(x)\, F\left(1+\frac{N+z}{2}, 1+\frac{N-z}{2}; \frac{3}{2}; -\sinh^2(x)\right),$$

with

$$c_{even} = \frac{2^z \Gamma(1/2)\, \Gamma(1-z)}{\Gamma\left(1+\frac{N-z}{2}\right) \Gamma\left(\frac{1-N-z}{2}\right)} = 2^{z-1} \frac{\Gamma\left(\frac{1+z+N}{2}\right) \Gamma\left(\frac{z-N}{2}\right)}{\Gamma\left(\frac{1}{2}\right) \Gamma(z)},$$

$$c_{odd} = \frac{2^z \Gamma(-1/2)\, \Gamma(1-z)}{\Gamma\left(\frac{1+N-z}{2}\right) \Gamma\left(\frac{-N-z}{2}\right)} = 2^{z-1} \frac{\Gamma\left(1+\frac{z+N}{2}\right) \Gamma\left(\frac{1+z-N}{2}\right)}{\Gamma\left(\frac{3}{2}\right) \Gamma(z)}.$$

Here we have used: (i) the hypergeometric representation for the associated Legendre function on the third line of Eq. (2.3), (ii) the quadratic hypergeometric $F(-, -; -; \xi) \to F(-, -; -; 4\xi(\xi-1)/(1-2\xi)^2)$ transformation formula [AS2, Eq. 15.3.29], and (iii) the linear hypergeometric $F(-, -; -; \xi) \to F(-, -; -; 1/\xi)$ transformation formula [GR, Eq. 9.132.2]. We have furthermore employed the reflection property $\Gamma(\xi)\,\Gamma(1-\xi) = \pi/\sin(\pi\xi)$ to rewrite the products of gamma functions appearing in the normalization constants c_{even} and c_{odd}. (At this point one also uses that N is an integer.)

From the above expressions one reads-off hypergeometric representations for the even and odd (in the variable x) component of the normalized associated Legendre function $\Psi_N(x, z)$. Division by $(2\cosh(x))^{N+1}$ and plugging in the combinatorial formula for $\Psi_N(x, z)$ from Proposition 4.1/Corollary 4.3 then entails the formulas stated by the proposition. Indeed, it is immediate from the combinatorial formula of Proposition 4.1/Corollary 4.3, combined with Eq. (4.5), that $\Psi_N(x, z)/(2\cosh(x))^{N+1} = \Phi_N^{(1)}(x, z)$. ∎

Proposition 5.2 (Case II: $H_+^{1,2N+3} = SO_0(2N+3, 1; \mathbb{R})/SO(2N+3; \mathbb{R})$). *Let $N \in \mathbb{N}^*$ and let $z \in \mathbb{C} \backslash \mathbb{N}$. Then one has that*

$$\frac{2^z \Gamma\left(\frac{1+z+N}{2}\right) \Gamma\left(1+\frac{N+z}{2}\right)}{2^{N+1} \Gamma(z)\, \Gamma(N+\frac{3}{2})}$$

$$\times F\left(\frac{N+1+z}{2}, \frac{N+1-z}{2}; N+\frac{3}{2}; -\sinh^2(x)\right)$$

$$= (\Phi_N^{(2)}(x, z) + \Phi_N^{(2)}(-x, z)),$$

where

$$\Phi_N^{(2)}(x, z) = \exp((z-N-1)\,x)$$

$$\times \frac{\sum_{J \subset \{1, ..., N\}} (-1)^{\sum_{j \in J} j} \prod_{j \in J} \left(\frac{z+j}{z-j}\right) \prod_{j \in J, k \notin J} \left|\frac{j+k}{j-k}\right| \exp\left(-2x \sum_{j \in J} j\right)}{\sum_{J \subset \{1, ..., N+1\}} (-1)^{\sum_{j \in J} j} \prod_{j \in J, k \notin J} \left|\frac{j+k}{j-k}\right| \exp\left(-2x \sum_{j \in J} j\right)}$$

$$= \exp((z-N-1)\,x) \frac{\sum_\lambda (-1)^{|\lambda|/2} c_\lambda(z)\, s_\lambda(e^{-x} \mathbf{1}_N)}{\sum_\lambda (-1)^{|\lambda|/2}\, s_\lambda(e^{-x} \mathbf{1}_{N+1})}.$$

(*Here the notation and the range of the summations is in correspondence with that of Proposition* 5.1 *and* $|\lambda|$ *denotes the weight of the partition* λ, *i.e., the sum of its parts.*)

Proof. The proof of the noncompact Riemannian case is very similar to that of the noncompact pseudo-Riemannian case of Proposition 5.1. On the one hand we deduce via hypergeometric representations for the associated Legendre function that

$$i^{-z} \Psi_N(x + i\pi/2, z) = i^{-z} \Gamma(1-z)\, P_N^z(\coth(x))$$

$$\overset{\text{Eq. 2.3}}{=} (e^x - e^{-x})^z\, F\left(N+1-z, -N-z; 1-z; \frac{1-\coth(x)}{2}\right)$$

$$\overset{\text{(i)}}{=} (e^x - e^{-x})^z F\left(\frac{N+1-z}{2}, \frac{-N-z}{2}; 1-z; -\frac{1}{\sinh^2(x)}\right)$$

$$\overset{\text{(ii)}}{=} c_{reg} \sinh^{N+1}(x)\, F\left(\frac{N+1+z}{2}, \frac{N+1-z}{2}; N+\frac{3}{2}; -\sinh^2(x)\right)$$

$$+ c_{sing} \sinh^{-N}(x)\, F\left(\frac{-N+z}{2}, \frac{-N-z}{2}; -N+\frac{1}{2}; -\sinh^2(x)\right),$$

with

$$c_{reg} = \frac{2^z \Gamma(1-z)\, \Gamma(-N-1/2)}{\Gamma\left(\frac{1-N-z}{2}\right) \Gamma\left(\frac{-N-z}{2}\right)} \overset{\text{(iii)}}{=} 2^{z-1} \frac{\Gamma\left(\frac{1+z+N}{2}\right) \Gamma\left(1+\frac{N+z}{2}\right)}{\Gamma(z)\, \Gamma\left(N+\frac{3}{2}\right)},$$

$$c_{sing} = \frac{2^z \Gamma(1-z)\, \Gamma(N+1/2)}{\Gamma\left(\frac{1+N-z}{2}\right) \Gamma\left(1+\frac{N-z}{2}\right)}$$

(where one uses: (i) the quadratic $F(-,-;-;\xi) \to F(-,-;-;4\xi(1-\xi))$ transformation [AS2, Eq. 15.3.30], (ii) the linear $F(-,-;-;\xi) \to F(-,-;-;1/\xi)$ transformation [GR, Eq. 9.132.2], and (iii) the reflection relation of the gamma function). On the other hand it is clear from the combinatorial representation of Proposition 4.1/Corollary 4.3 that $i^{-z}\Psi_N(x+i\pi/2, z) = (2\sinh(x))^{N+1}\, \Phi_N^{(2)}(x, z)$.

The formula of the proposition thus follows upon division by $(2\sinh(x))^{N+1}$ and selection of the regular (i.e., even) component in x. ∎

PROPOSITION 5.3 (Case III: $S^{2N+3} = SO(2N+4;\mathbb{R})/SO(2N+3;\mathbb{R})$). *Let $N \in \mathbb{N}^*$ and let $n \in \mathbb{N}$. Then one has that*

$$\frac{(2N+2)_n}{(N+1)_n} F\left(-n, n+2(N+1); N+\frac{3}{2}; \sin^2\left(\frac{x}{2}\right)\right)$$
$$= (\Phi_N^{(3)}(x, n) + \Phi_N^{(3)}(-x, n)),$$

where

$$\Phi_N^{(3)}(x, n) = \exp(inx)$$
$$\times \frac{\sum_{J \subset \{1, ..., N\}} (-1)^{\sum_{j\in J} j} \prod_{j \in J} \left(\frac{n+N+1+j}{n+N+1-j}\right) \prod_{j\in J, k \notin J} \left|\frac{j+k}{j-k}\right| \exp\left(-2ix \sum_{j\in J} j\right)}{\sum_{J \subset \{1, ..., N+1\}} (-1)^{\sum_{j\in J} j} \prod_{j\in J, k\notin J} \left|\frac{j+k}{j-k}\right| \exp\left(-2ix\sum_{j\in J} j\right)}$$
$$= \exp(inx) \frac{\sum_\lambda (-1)^{|\lambda|/2}\, c_\lambda(n+N+1)\, s_\lambda(e^{-ix}\mathbf{1}_N)}{\sum_\lambda (-1)^{|\lambda|/2}\, s_\lambda(e^{-ix}\mathbf{1}_{N+1})}.$$

(*Here the notation and the range of the summations is in correspondence with that of Proposition* 5.1 *and $|\lambda|$ denotes the weight of the partition λ, i.e., the* *sum of its parts. Moreover, the Pochhammer symbol* $(a)_m$ *stands for the quotient* $\Gamma(a+m)/\Gamma(a)$.)

Proof. The compact Riemannian case follows from the noncompact Riemannian case of Proposition 5.2 by the substitution $x \to ix$ and $z \to n+N+1$. To arrive at the stated hypergeometric representation for the l.h.s., one performs the quadratic transformation (cf. [AS2, Eq. 15.3.30])

$$F\left(-\frac{n}{2}, N+1+\frac{n}{2}; N+\frac{3}{2}; \sin^2(x)\right) = F\left(-n, 2(N+1)+n; N+\frac{3}{2}; \sin^2\left(\frac{x}{2}\right)\right),$$

and one rewrites the normalizing gamma factors in front as

$$2^n \frac{\Gamma(N+1+n/2)\,\Gamma(N+(n+3)/2)}{\Gamma(N+1+n)\,\Gamma(N+3/2)} = \frac{(n+N+1)_{N+1}}{(N+1)_{N+1}} = \frac{(2N+2)_n}{(N+1)_n}. \quad \blacksquare$$

Remark 5.1. Although formulated only for positive N for reasons of convenience, the statements of Propositions 5.1–5.3 in fact still hold true in the trivial case $N=0$ (cf. Remark 4.1). As a matter of fact, the combinatorial formulas reduce in this situation to well-known evaluation formulas for certain elementary Gauss hypergeometric series. Specifically, for $N=0$ Proposition 5.1 specializes to the identities

$$F\left(\frac{1+z}{2}, \frac{1-z}{2}; \frac{1}{2}; -\sinh^2(x)\right) = \frac{\cosh(zx)}{\cosh(x)}$$

and

$$z\sinh(x)\, F\left(1+\frac{z}{2}, 1-\frac{z}{2}; \frac{3}{2}; -\sinh^2(x)\right) = \frac{\sinh(zx)}{\cosh(x)} \tag{5.7b}$$

(cf. [AS2, Eq. 15.1.18] and [AS2, Eq. 15.1.16]), Proposition 5.2 specializes to the identity

$$zF\left(\frac{1+z}{2}, \frac{1-z}{2}; \frac{3}{2}; -\sinh^2(x)\right) = \frac{\sinh(zx)}{\sinh(x)} \tag{5.8}$$

(cf. [AS2, Eq. 15.1.15]), and Proposition 5.3 specializes to the indentity

$$(n+1)\, F\left(-n, n+2; \frac{3}{2}; \sin^2\left(\frac{x}{2}\right)\right) = \frac{\sin((n+1)\,x)}{\sin(x)} \tag{5.9}$$

(cf. [AS2, Eq. 15.1.16]). The elementary functions in question are zonal spherical functions on the symmetric spaces $H^{3,1}=SO_0(3,1;\mathbb{R})/SO_0(2,1;\mathbb{R})$ (Eqs. (5.7a), (5.7b)), $H_+^{1,3}=SO_0(3,1;\mathbb{R})/SO(3;\mathbb{R})$ (Eq. (5.8)), and $S^3=SO(4;\mathbb{R})/SO(3;\mathbb{R})$ (Eq. (5.9)), respectively.

Remark 5.2. In the compact Riemannian situation of the sphere $S^{2N+3}=SO(2N+4;\mathbb{R})/SO(2N+3;\mathbb{R})$ (Case III), the zonal spherical functions $\phi_N^{(3)}(x,n)$ (5.6) boil—up to a normalization factor—down to ultraspherical (or Gegenbauer) polynomials [AS2, GR]

$$C_n^\lambda(\cos(x))=\frac{(2\lambda)_n}{n!}\,F\left(-n,n+2\lambda;\lambda+\frac{1}{2};\sin^2\left(\frac{x}{2}\right)\right),\qquad \lambda=N+1. \tag{5.10}$$

The trivial case $N=0$ (cf. Remark 5.1) corresponds from this viewpoint to the specialization of the ultraspherical polynomials to Chebyshef polynomials of the second kind $U_n(\cos(x))=\sin((n+1)\,x)/\sin(x)$ (cf. Eq. 5.9). In other words, Proposition 5.3 may be looked upon as a combinatorial formula for the ultraspherical polynomials $C_n^\lambda(\cos(x))$, corresponding to positive integer values of the parameter λ. Notice, however, that the individual building blocks for the r.h.s. of the combinatorial formula, viz. the functions $\Phi_N^{(3)}(\pm x,n)$, have a singularity at $x=0\pmod{\pi}$. (This is because the denominators factorize as $(1-\exp(\mp 2ix))^{(N+1)(N+2)/2}$ (cf. Remark 4.3) and—apart from the overall factor $\exp(\pm inx)$—the numerators are only of degree $N(N+1)$ in $\exp(\mp 2ix)$.) In particular, the functions $\Phi_N^{(3)}(\pm x,n)$ are *not* Laurent polynomials in $\exp(ix)$, even though the even combination $\Phi_N^{(3)}(x,n)+\Phi_N^{(3)}(-x,n)$ is so (since it is a polynomial in $\cos(x)$). (This state of affairs is of course in correspondence with the fact that the differential equation for the zonal spherical function in (5.3) has a regular singularity at $x=0\pmod{\pi}$.)

Remark 5.3. In Propositions 5.1 and 5.2 the zonal spherical functions are normalized such that they have an asymptotics of the form

$$e^{(z-N-1)\,x}\pm e^{(N+1-z)\,x}\prod_{j=1}^{N}\left(\frac{z+j}{z-j}\right)\qquad\text{for}\quad x\to\infty. \tag{5.11}$$

(Here the plus sign corresponds to the even cases and the minus sign corresponds to the odd case.) Notice that these asymptotics are immediate from the combinatorial representations on the r.h.s.

In the case of Proposition 5.3 the zonal spherical functions amount to ultraspherical polynomials in $\cos(x)$ in view of the previous remark. The normalization adopted by the proposition corresponds in this connection to that of ultraspherical polynomials that are monic in $e^{ix}+e^{-ix}$ (as $(2N+2)_n/(N+1)_n=2^{2n}\,(N+3/2)_n/(2N+2+n)_n$).

Remark 5.4. It is instructive to further emphasize the close connection between the analysis of zonal spherical functions on hyperboloids and spheres and the study of exactly solvable one-dimensional quantum-mechanical models. To this end we consider the following three closely related Pöschl–Teller-type Schrödinger equations (on the line, half-line, and finite interval, respectively):

$$\left(\frac{d^2}{dx^2}+\frac{N(N+1)}{\cosh^2(x)}-z^2\right)\psi_N^{(1)}(x,z)=0, \qquad -\infty<x<\infty, \tag{5.12}$$

$$\left(\frac{d^2}{dx^2}-\frac{N(N+1)}{\sinh^2(x)}-z^2\right)\psi_N^{(2)}(x,z)=0, \qquad 0<x<\infty, \tag{5.13}$$

$$\left(\frac{d^2}{dx^2}-\frac{N(N+1)}{\sin^2(x)}+(n+N+1)^2\right)\psi_N^{(3)}(x,n)=0, \qquad 0<x<\pi. \tag{5.14}$$

It is not difficult to infer that the substitutions

$$\psi_N^{(1)}(x,z)=\cosh^{N+1}(x)\,\phi_N^{(1)}(x,z), \tag{5.15}$$

$$\psi_N^{(2)}(x,z)=\sinh^{N+1}(x)\,\phi_N^{(2)}(x,z), \tag{5.16}$$

$$\psi_N^{(3)}(x,n)=\sin^{N+1}(x)\,\phi_N^{(3)}(x,n), \tag{5.17}$$

transform the above Schrödinger equations into the second-order differential equations for the zonal-spherical functions of Eqs. (5.1), (5.2), and (5.3), respectively. Connections of this kind between the harmonic analysis on symmetric spaces of simple Lie groups and the study of certain (integrable) quantum-mechanical systems are well known and generalize in fact to the situation of higher rank symmetric spaces [OP]. For us it means, in particular, that Propositions 5.1, 5.2, and 5.3 give rise to combinatorial formulas for the wave functions of the Pöschl–Teller eigenvalue problems (5.12), (5.13), and (5.14). Specifically, for Eq. (5.12) multiplication of the zonal spherical functions from Proposition 5.1 by $(2\cosh(x))^{N+1}$ brings us back to the even/odd solutions $\Psi_N^{(1)}(x,z)\pm\Psi_N^{(1)}(-x,z)$ with $\Psi^{(1)}(x,z)=\Psi(x,z)$ taken from Proposition 4.1/Corollary 4.3. Similarly, for Eqs. (5.13) and (5.14) one obtains—by multiplication of the zonal spherical functions of Propositions 5.2 and 5.3 by $(2\sinh(x))^{N+1}$ and $(2\sin(x))^{N+1}$ —regular solutions of the form $\Psi_N^{(2)}(x,z)+\Psi_N^{(2)}(-x,z)$ and $\Psi_N^{(3)}(x,n)+\Psi_N^{(3)}(-x,n)$, where $\Psi_N^{(2)}(x,z)$ and $\Psi_N^{(3)}(x,n)$ are obtained from $\Phi_N^{(2)}(x,z)$ (cf. Proposition 5.2) and $\Phi_N^{(3)}(x,n)$ (cf. Proposition 5.3) by the replacements $\exp((z-N-1)\,x)\to\exp(zx)$ and $\exp(inx)\to\exp(i(n+N+1)\,x)$, respectively, and the summation in the denominator becomes over index sets J with indices from $1, ..., N$ instead of $1, ..., N+1$. Correspondingly, in the denominator of the Schur-function representation

$s_\lambda(e^{-(i)x}\mathbf{1}_{N+1}) \to s_\lambda(e^{-(i)x}\mathbf{1}_N)$ and the summation becomes over partitions with N instead of $N+1$ parts, i.e., the number of variables for the Schur functions in the denominator decreases by one. (In other words, $\Psi^{(2)}(x,z) = i^{-z}\Psi^{(1)}(x+i\pi/2, z)$ and $\Psi^{(3)}(x,n) = \Psi^{(2)}(ix, n+N+1)$ with $\Psi^{(1)}(x,z) = \Psi(x,z)$ taken from Proposition 4.1/Corollary 4.3.)

APPENDIX: PROOF OF THE SATO FORMULA USING INVERSE SCATTERING THEORY

In this Appendix we reconstruct the Jost solution from the spectral data of a one-dimensional Schrödinger equation with a potential of Bargmann type (i.e., a reflectionless Schwartz class potential). The reconstruction culminates in the Sato formula for the reflectionless Jost function (cf. Eq. (3.5)), which was used in Section 4. to arrive at the combinatorial representations for the integer-degree associated Legendre functions.

Our starting point is a fundamental theorem from the inverse-scattering theory for one-dimensional Schrödinger equations with Schwartz class potentials. The theorem states that—for a general Schrödinger equation with $u(x) \in \mathscr{S}(\mathbb{R})$ say[2]—the potential $u(x)$ and the Jost function $\Psi_{jost}(x,z)$ can be expressed in terms of the spectral data in the following way [SCM, DT, AS1, NMPZ]

$$u(x) = 2\frac{\mathrm{d}}{\mathrm{d}x} K(x,x), \tag{A.1}$$

$$\Psi_{jost}(x,z) = \exp(zx) + \int_x^\infty K(x,\xi)\exp(z\xi)\,\mathrm{d}\xi \qquad (\mathrm{Re}(z)=0), \tag{A.2}$$

where $K(x,y)$ is the (unique) solution to the *Gelfand–Levitan–Marchenko integral equation*

$$K(x,y) + F(x+y) + \int_x^\infty K(x,\xi)\,F(\xi+y)\,\mathrm{d}\xi = 0 \qquad (x<y) \tag{A.3a}$$

with

$$F(x) = \sum_{j=1}^N \nu_j e^{-\kappa_j x} + \frac{1}{2\pi}\int_{-\infty}^\infty r(i\zeta)\,e^{ix\zeta}\,\mathrm{d}\zeta. \tag{A.3b}$$

Here $-\kappa_1, ..., -\kappa_N(<0)$ and $\nu_1, ..., \nu_N(>0)$ are the spectral values and normalization constants of the bound-state eigenfunctions, respectively,

[2] It is in fact possible to relax the conditions on the decay of the potential $u(x)$ considerably to smooth real potentials for which the integral $\int_{-\infty}^{\infty}(1+x^2)\,u(x)\,\mathrm{d}x$ exists [DT].

and $r(z)$ denotes the reflection coefficient for the potential under consideration (cf. Section 3). In the reflectionless situation with $r(z)=0$, the Gelfand–Levitan–Marchenko equation becomes separable and reduces to a finite-dimensional linear algebraic system [SCM, DT, AS1, NMPZ]. Indeed, for $F(x)=\sum_{j=1}^{N} v_j e^{-\kappa_j x}$ substitution of the Ansatz

$$K(x, y)=\sum_{j=1}^{N} K_j(x)\, e^{-\kappa_j y} \tag{A.4}$$

turns the Gelfand–Levitan–Marchenko equation into the linear system

$$K_j(x)+v_j \sum_{k=1}^{N} \frac{e^{-(\kappa_j+\kappa_k)\,x}}{\kappa_j+\kappa_k} K_k(x)=-v_j e^{-\kappa_j x}, \qquad j=1, ..., N. \tag{A.5}$$

The solution of the linear system is given by

$$K_j(x)=\frac{\det \mathbf{A}^{(j)}(x)}{\det \mathbf{A}(x)}, \qquad j=1, ..., N, \tag{A.6}$$

where $\mathbf{A}(x)$ and $\mathbf{A}^{(j)}(x)$ denote the matrices

$$\mathbf{A}(x)=\begin{bmatrix} 1+v_1\dfrac{e^{-2\kappa_1 x}}{2\kappa_1} & \cdots & v_1\dfrac{e^{-(\kappa_1+\kappa_j)\,x}}{\kappa_1+\kappa_j} & \cdots & v_1\dfrac{e^{-(\kappa_1+\kappa_N)\,x}}{\kappa_1+\kappa_N} \\ \vdots & \ddots & \vdots & & \vdots \\ v_j\dfrac{e^{-(\kappa_j+\kappa_1)\,x}}{\kappa_j+\kappa_1} & \cdots & 1+v_j\dfrac{e^{-2\kappa_j x}}{2\kappa_j} & \cdots & v_j\dfrac{e^{-(\kappa_j+\kappa_N)\,x}}{\kappa_j+\kappa_N} \\ \vdots & & \vdots & \ddots & \vdots \\ v_N\dfrac{e^{-(\kappa_N+\kappa_1)\,x}}{\kappa_N+\kappa_1} & \cdots & v_N\dfrac{e^{-(\kappa_N+\kappa_j)\,x}}{\kappa_N+\kappa_j} & \cdots & 1+v_N\dfrac{e^{-2\kappa_N x}}{2\kappa_N} \end{bmatrix} \tag{A.7}$$

and

$$\mathbf{A}^{(j)}(x)=\begin{bmatrix} 1+v_1\dfrac{e^{-2\kappa_1 x}}{2\kappa_1} & \cdots & -v_1 e^{-\kappa_1 x} & \cdots & v_1\dfrac{e^{-(\kappa_1+\kappa_N)\,x}}{\kappa_1+\kappa_N} \\ \vdots & \ddots & \vdots & & \vdots \\ v_j\dfrac{e^{-(\kappa_j+\kappa_1)\,x}}{\kappa_j+\kappa_1} & \cdots & -v_j\, e^{-\kappa_j x} & \cdots & v_j\dfrac{e^{-(\kappa_j+\kappa_N)\,x}}{\kappa_j+\kappa_N} \\ \vdots & & \vdots & \ddots & \vdots \\ v_N\dfrac{e^{-(\kappa_N+\kappa_1)\,x}}{\kappa_N+\kappa_1} & \cdots & -v_N\, e^{-\kappa_N x} & \cdots & 1+v_N\dfrac{e^{-2\kappa_N x}}{2\kappa_N} \end{bmatrix} \tag{A.8}$$

(i.e., $\mathbf{A}^{(j)}(x)$ is obtained from $\mathbf{A}(x)$ by replacing the jth column by $[\mathbf{A}^{(j)}(x)]_{k,j} = -\nu_k e^{-\kappa_k x}$, $k=1, ..., N$).

Substitution of the kernel $K(x, y)$ 6.5, with coefficients $K_j(x)$ of the form (A.6), into Eq. (A.1) produces the Hirota formula for the potential $u(x)$ [SCM, AS1, NMPZ]

$$u(x) = 2\,\frac{d^2}{dx^2}\log \tau(x),$$

$$\tau(x) = \det \mathbf{A}(x) = \sum_{J \subset \{1, ..., N\}} \prod_{j \in J} \frac{\nu_j}{2\kappa_j} \prod_{j,k \in J;\, j<k} \left(\frac{\kappa_j - \kappa_k}{\kappa_j + \kappa_k}\right)^2 \exp\left(-2 \sum_{j \in J} \kappa_j x\right).$$

Indeed, it is clear from the explicit expressions for $\mathbf{A}(x)$ and $\mathbf{A}^{(j)}(x)$ that the function $K(x, x) = \sum_{j=1}^{N} e^{-\kappa_j x} \det \mathbf{A}^{(j)}(x)/\det \mathbf{A}(x)$ appearing on the r.h.s. of (A.1) is equal to the logarithmic derivative of $\det \mathbf{A}(x)$. The evaluation of the determinant of $\mathbf{A}(x)$ moreover hinges on the Cauchy determinant formula [M, p. 67]

$$\det\left[\frac{1}{x_j + y_k}\right]_{1 \leqslant j,k \leqslant N} = \frac{\prod_{1 \leqslant j<k \leqslant N} (x_j - x_k)(y_j - y_k)}{\prod_{1 \leqslant j,k \leqslant N} (x_j + y_k)}. \tag{A.9}$$

Specifically, the matrix $\mathbf{A}(x)$ (A.7) is a sum of the identity matrix and a product of the form $\mathbf{NDCD}$ with $N = \mathrm{diag}(\nu_1, ..., \nu_N)$, $\mathbf{D} = \mathrm{diag}(e^{-\kappa_1 x}, ..., e^{-\kappa_N x})$ and $\mathbf{C} = [(\kappa_j + \kappa_k)^{-1}]_{1 \leqslant j,k \leqslant N}$. Hence, $\det \mathbf{A}(x)$ is equal to the sum of all principal minors of the matrix $\mathbf{NDCD}$. Evaluation of these minors by means of the Cauchy determinant in (A.9) then entails the above Hirota formula.

To compute the corresponding Jost eigenfunction one substitutes the finite-dimensional kernel $K(x, y)$ (A.4), (A.6) in the formula of (A.2). Performing the integration in the second term on the r.h.s. of (A.2). results—after some further manipulations—in Sato's formula for the reflectionless Jost function:

$$\Psi_{jost}(x, z) = \exp(zx)\left(1 - \sum_{j=1}^{N} \frac{\det \mathbf{A}^{(j)}(x)}{\det \mathbf{A}(x)} \frac{e^{-\kappa_j x}}{(z - \kappa_j)}\right)$$

$$= \exp(zx)\,\frac{\det \mathbf{B}(x, z)}{\det \mathbf{A}(x)}$$

$$= \exp(zx)\,\frac{\displaystyle\sum_{J \subset \{1, ..., N\}} \prod_{j \in J} \frac{\nu_j}{2\kappa_j}\left(\frac{z + \kappa_j}{z - \kappa_j}\right) \prod_{j,k \in J;\, j<k} \left(\frac{\kappa_j - \kappa_k}{\kappa_j + \kappa_k}\right)^2 \exp\left(-2\sum_{j \in J} \kappa_j x\right)}{\displaystyle\sum_{J \subset \{1, ..., N\}} \prod_{j \in J} \frac{\nu_j}{2\kappa_j} \prod_{j,k \in J;\, j<k} \left(\frac{\kappa_j - \kappa_k}{\kappa_j + \kappa_k}\right)^2 \exp\left(-2\sum_{j \in J} \kappa_j x\right)},$$

where $\mathbf{B}(x, z)$ denotes the matrix

$$\mathbf{B}(x, z) =$$

$$\begin{bmatrix} 1+\frac{\nu_1}{2\kappa_1}\left(\frac{z+\kappa_1}{z-\kappa_1}\right)e^{-2\kappa_1 x} & \cdots & \frac{\nu_1}{\kappa_1+\kappa_N}\left(\frac{z+\kappa_1}{z-\kappa_1}\right)e^{-(\kappa_1+\kappa_N)x} \\ \vdots & \ddots & \vdots \\ \frac{\nu_N}{\kappa_N+\kappa_1}\left(\frac{z+\kappa_N}{z-\kappa_N}\right)e^{-(\kappa_N+\kappa_1)x} & \cdots & 1+\frac{\nu_N}{2\kappa_N}\left(\frac{z+\kappa_N}{z-\kappa_N}\right)e^{-2\kappa_N x} \end{bmatrix}$$

(i.e., $\mathbf{B}(x, z)$ is obtained from $\mathbf{A}(x)$ (A.7) by means of the substitution $\nu_j \to \nu_j(z+\kappa_j)/(z-\kappa_j)$). Indeed, the formula on the first line is immediate from Eq. (A.2) with $K(x, y)$ taken from Eqs. (A.4), (A.6). To pass to the formula on the second line one uses the following pole expansion for $\det \mathbf{B}(x, z)$

$$\det \mathbf{B}(x, z) = \det \mathbf{B}_\infty(x) + \sum_{j=1}^{N} \frac{\det \mathbf{B}_j(x)}{z-\kappa_j}, \tag{A.10}$$

where $\mathbf{B}_\infty(x) = \lim_{z\to\infty} \mathbf{B}(x, z) = \mathbf{A}(x)$ and $B_j(x)$ is the matrix obtained from $B(x_j z)$ via the substitution $z = \kappa_j$, after multiplication of the jth row by $z-\kappa_j$. Subtracting $e^{(\kappa_j-\kappa_k)x}$ times jth column of the residue matrix $\mathbf{B}_j(x)$ from its kth column, for $k = 1, ..., N$, $k \neq j$, and multiplying the resulting matrix from the left by $\mathrm{diag}(\kappa_j-\kappa_1, ..., 1, ..., \kappa_j-\kappa_N)$ and from the right by $\mathrm{diag}((\kappa_j-\kappa_1)^{-1}, ..., 1, ..., (\kappa_j-\kappa_N)^{-1})$ (where the unit is in the jth slot) yields a matrix that differs from the matrix $\mathbf{A}^{(j)}(x)$ by multiplication of the jth column by $-e^{-\kappa_j x}$ (cf. Eq. (A.8)). Hence, we conclude that the expressions for $\Psi_{jost}(x, z)$ on the first and second line indeed coincide. The Sato formula on the third line then follows from the second line via the explicit evaluation of $\det \mathbf{A}(x)$ and $\det \mathbf{B}(x, z)$ by means of the Cauchy determinant formula (cf. above).

Notice that initially in Eq. (A.2). the spectral parameter z was chosen on the imaginary axis (in order to guarantee that the integrals converge). It is clear by analyticity, however, that the resulting Sato formula then in fact holds (i.e., solves the Schrödinger equation with corresponding Hirota potential) for $z \in \mathbb{C}\setminus\{\kappa_1, ..., \kappa_N\}$.

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