On the proof of (4.13) and (4.14) in Butzer & Koornwinder (2019)

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This note gives a proof of formulas (4.13) and (4.14) in [1], which are formulas (5.4) and (5.5) in Meixner's paper [2]. The proof given here has some more details than the proof in [2].

Assume that $P_{-1}(x) = 0$. The following equations are given, see [1, (4.3), (4.5) and (4.12)]:

$$t(D) = D + a_2 D^2 + a_3 D^3 + \cdots \,, \tag{1}$$

$$t(D)P_n(x) = nP_{n-1}(x), \qquad n = 0, 1, 2, \ldots, \tag{2}$$

$$P_{n+1}(x) = (x + l_{n+1})P_n(x) + k_{n+1}P_{n-1}(x), \qquad n = 0, 1, 2, \ldots \,. \tag{3}$$

Note that, for a C^∞-function f,

$$D^n(xf(x)) = \sum_{k=0}^{n} \binom{n}{k} D^k x\, D^{n-k} f(x) = xD^n f(x) + nD^{n-1} f(x).$$

Hence

$$\begin{aligned} t(D)(xP_n(x)) &= x\, t(D)P_n(x) + (1 + 2a_2 D + 3a_3 D^2 + \cdots)P_n(x) \\ &= nP_{n-1}(x) + t'(D)P_n(x). \end{aligned} \tag{4}$$

Now apply $t(D)$ to (3) and use (2) and (4):

$$(n+1)P_n(x) = t'(D)P_n(x) + n(x + l_{n+1})P_{n-1}(x) + (n-1)k_{n+1}P_{n-2}(x), \quad n = 1, 2, \ldots \,. \tag{5}$$

From (3) we get

$$nP_n(x) = n(x + l_n)P_{n-1}(x) + nk_n P_{n-2}(x), \quad n = 1, 2, \ldots \,. \tag{6}$$

Subtract (6) from (5). Then

$$(1 - t'(D))P_n(x) = n(l_{n+1} - l_n)P_{n-1}(x) + ((n-1)k_{n+1} - nk_n)P_{n-2}(x), \quad n = 1, 2, \ldots \,. \tag{7}$$

Apply $t(D)$ to (7) and use (2). Then, for $n = 2, 3, \ldots$,

$$n(1 - t'(D))P_{n-1}(x) = n(n-1)(l_{n+1} - l_n)P_{n-2}(x) + (n-2)((n-1)k_{n+1} - nk_n)P_{n-3}(x).$$

Hence, for $n = 1, 2, \ldots$,

$$(1 - t'(D))P_n(x) = n(l_{n+2} - l_{n+1})P_{n-1}(x) + (n+1)^{-1}(n-1)(nk_{n+2} - (n+1)k_{n+1})P_{n-2}(x). \tag{8}$$

From (7) and (8) we get

$$\begin{aligned} \lambda &= l_{n+2} - l_{n+1} = l_{n+1} - l_n, & n &= 1, 2, \ldots \,, \\ \kappa &= (n+1)^{-1}k_{n+2} - n^{-1}k_{n+1} = n^{-1}k_{n+1} - (n-1)^{-1}k_n, & n &= 2, 3 \ldots \,, \end{aligned}$$

with λ and κ independent of n. Iteration gives

$$l_{n+1} = l_1 + n\lambda, \qquad n^{-1}k_{n+1} = k_2 + (n-1)\kappa.$$

Hence we can rewrite (3) as

$$P_{n+1}(x) = (x + l_1 + n\lambda)P_n(x) + n(k_2 + (n-1)\kappa)P_{n-1}(x), \qquad n = 0, 1, 2, \dots \,. \tag{9}$$

We can also rewrite (7) as

$$(1 - t'(D))P_n(x) = n\lambda P_{n-1}(x) + n(n-1)\kappa P_{n-2}(x), \quad n = 1, 2, \dots \,.$$

In combination with (2) this gives

$$(1 - t'(D))P_n(x) = \lambda t(D)P_n(x) + \kappa t(D)^2 P_n(x), \quad n = 1, 2, \dots \,. \tag{10}$$

Since $1 - t'(D)$ is a polynomial in D without constant term, (10) also holds for $n = 0$. Hence

$$(1 - t'(D))f(x) = \lambda t(D)f(x) + \kappa t(D)^2 f(x)$$

for any polynomial $f(x)$. Therefore

$$1 - t'(D) = \lambda t(D) + \kappa t(D)^2.$$

So the power series $t(u)$ satisfies the differential equation

$$t'(u) = 1 - \lambda t(u) - \kappa t(u)^2. \tag{11}$$

References

[1] P. L. Butzer and T. H. Koornwinder, *Josef Meixner: his life and his orthogonal polynomials*, Indag. Math. (N.S.) 30 (2019), 250–264.

[2] J.Meixner, *Orthogonale Polynomsysteme mit einer besonderen Gestalt der erzeugenden Funktion*, J. London Math. Soc. 9 (1934), 6–13.