Bott Periodicity & Recursion

0^{th}: $a \in \mathbb{R}$

$$\downarrow$$

1^{st} $a + bi \in \mathbb{C} \longleftrightarrow \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

$$\in M_2(\mathbb{R})$$

How?

$$J_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$A_1 = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

demand: $A_1 J_1 = J_1 A_1$

$$\Rightarrow \quad d = a, \quad c = -b$$

$$A_1 = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

$$A_2 = \left(\begin{array}{cc|cc} a_1 & -b_1 & a_3 & -b_3 \\ b_1 & a_1 & b_3 & a_3 \\ \hline a_2 & -b_2 & a_4 & -b_4 \\ b_2 & a_2 & b_4 & a_4 \end{array}\right) \in M_2(\mathbb{C})$$

$$P_{23} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$P_{23}^t P_{23} = I \qquad P_{23}^2 = I$$

$$P_{23}^t = P_{23}$$

Demand

$P_{23} A_2 P_{23}$ commute with $\begin{pmatrix} J_1 & 0 \\ 0 & -J_1 \end{pmatrix}$

$$\left(\begin{array}{cc|cc} a_1 & a_3 & -b_1 & -b_3 \\ a_2 & a_4 & -b_2 & -b_4 \\ \hline b_1 & b_3 & a_1 & a_3 \\ b_2 & b_4 & a_2 & a_4 \end{array}\right)$$

commute with

$$J_2' = \left(\begin{array}{c|c} J_1 & 0 \\ \hline 0 & -J_1 \end{array}\right) \qquad J_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$J_1^{\exp} = \left(\begin{array}{cc|cc} 0 & -1 & & \\ 1 & 0 & \multicolumn{2}{c}{0} \\ \hline & & 0 & -1 \\ \multicolumn{2}{c|}{0} & 1 & 0 \end{array}\right) \ldots$$

$$J_2' = \left(\begin{array}{cc|cc} 0 & \cdot 1 & & \\ 1 & 0 & \multicolumn{2}{c}{0} \\ \hline & & 0 & 1 \\ \multicolumn{2}{c|}{0} & -1 & 0 \end{array}\right) \ldots$$

$P_{23} A_2 P_{23}$ commutes with J_2'

$\Longleftrightarrow$

A_2 commutes with $P_{23} J_2' P_{23}$

$= J_3$

$$J_2 = \left(\begin{array}{cc|cc} 0 & & -1 & 0 \\ & & 0 & 1 \\ \hline 1 & 0 & & \\ 0 & -1 & & 0 \end{array}\right)$$

$$A_2 J_2 = J_2 A_2$$

$\Longleftrightarrow$

$$A_2 = \left(\begin{array}{cc|cc} a_1 & -a_2 & -b_1 & -b_2 \\ a_2 & a_1 & b_2 & -b_1 \\ \hline b_1 & -b_2 & a_1 & a_2 \\ b_2 & b_1 & -a_2 & a_1 \end{array}\right) = \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix} \in \mathbb{H}$$

$\begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix} \leftrightarrow$ right mult by $z + wj$

$A_2 \in \mathbb{H}$

<u>recursion</u>

$J_{j-1} = \begin{pmatrix} 0 & -K_{j-1}^t \\ K_{j-1} & 0 \end{pmatrix}$ K_{j-1} signed perm matrix equal # of $1, -1$

$J_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $K_1 = 1$

$$J_2 = \left(\begin{array}{cc|cc} & & -1 & 0 \\ \multicolumn{2}{c|}{0} & 0 & 1 \\ \hline 1 & 0 & & \\ 0 & -1 & \multicolumn{2}{c}{0} \end{array}\right) \qquad K_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$J_j = \left(\begin{array}{c|c} 0 & \begin{array}{c|c} -K_{j-1}^t & 0 \\ \hline 0 & K_{j-1}^t \end{array} \\ \hline \begin{array}{c|c} K_{j-1} & 0 \\ \hline 0 & -K_{j-1} \end{array} & 0 \end{array}\right)$$

show symm orthog

$$K_j = \begin{pmatrix} K_{j-1} & 0 \\ 0 & -K_{j-1} \end{pmatrix}$$ orthog signed perm

equal 1, -1

$K_1 = 1$

$K_2 = 1, -1$

$K_3 = 1, -1, -1, 1$

$K_4 = 1, -1, -1, 1, -1, 1, 1, -1$

$K_5 = 1, -1, -1, 1, -1, 1, 1, -1, -1, 1, 1, -1, 1, -1, -1, 1$

$$J_3 = \left(\begin{array}{c|c} 0 & \begin{matrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & -1 \end{matrix} \\ \hline \begin{matrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & 1 \end{matrix} & 0 \end{array}\right)$$

$$A_0 = (a) \in M_1(\mathbb{R})$$

$$\bar{A}_0 = A_0$$

$$A_1 = \begin{pmatrix} A_{01} & -\bar{A}_{02} \\ A_{02} & \bar{A}_{01} \end{pmatrix}$$

$$= \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

recursive defn of conj.

$$B_j = \begin{pmatrix} a_{j-1} & c_{j-1} \\ b_{j-1} & d_{j-1} \end{pmatrix}$$

$$\bar{B}_j = \begin{pmatrix} \bar{a}_{j-1} & -\bar{c}_{j-1} \\ \bar{b}_{j-1} & \bar{d}_{j-1} \end{pmatrix}$$

recursively we derive A_j from A_{j-1} by demanding $A_1 J_j = J_j A_j$

$$A_j = \begin{pmatrix} A_{j-1,1} & -\overline{A_{j-1,2}} \\ A_{j-1,2} & \overline{A_{j-1,1}} \end{pmatrix}$$

$A_0 = a$

$A_1 = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}$ on $\mathbb{R}$ $\bar{a} = a$

$$= \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

$$A_2 = \begin{pmatrix} A_{11} & -\bar{A}_{12} \\ A_{12} & \bar{A}_{11} \end{pmatrix}$$ $\bar{A}$ mean complex conj

$$= \begin{pmatrix} z & -\bar{\omega} \\ \omega & \bar{z} \end{pmatrix}$$

$$\bar{A}_2 = \begin{pmatrix} \bar{z} & -\overline{(-\bar{\omega})} \\ -\bar{\omega} & \bar{\bar{z}} \end{pmatrix}$$

$$= \begin{pmatrix} \bar{z} & \omega \\ -\bar{\omega} & z \end{pmatrix}$$ quat. conj

$$\longleftrightarrow \quad \bar{z} + (-\bar{\omega})j$$

$$z + \omega j = a + bi + (c + di)j$$

$$\overline{z + \omega j} = \bar{z} + (-\bar{\omega})j$$
$$= a + (-b)i + ((-c) + di)j$$

alg hom on $\mathbb{H}$

$$A_3 = \begin{pmatrix} q_1 & -\bar{q}_2 \\ q_2 & \bar{q}_1 \end{pmatrix} \qquad A_4 = \begin{pmatrix} A_{31} & -\bar{A}_{32} \\ A_{32} & \bar{A}_{31} \end{pmatrix}$$