Spreading of Wave function in Free Space (Dispersion / Smearing)

Basic Setup

$$L^2(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{C} \;\middle|\; \int_{-\infty}^{\infty} |f(x)|^2 dx < \infty \right\}$$

$f, g \in L^2(\mathbb{R})$ $\quad \langle f, g \rangle = \int_{-\infty}^{\infty} f(x)\, \overline{g(x)}\, dx$ "inner product"

"State" or "pre-probability" or "wavefunction" of a quantum system ("particle")

$$\psi_t(x) \in L^2(\mathbb{R})$$

time variable (t), spatial variable (x)

"Observables" are self-adjoint opr

$$A \in \mathcal{S}(L^2(\mathbb{R}))$$

i.e. $A^* = A$ $\quad \langle Af, g \rangle = \langle f, A^* g \rangle$

Born Rule: connection between A and measurement

average value of A $\quad E(A)_t = \langle A\psi_t, \psi_t \rangle = \int_{-\infty}^{\infty} (A\psi_t(x))\, \overline{\psi_t(x)}\, dx$

Time evolution of ψ_t: Schrodinger Eq

$$\boxed{i\hbar \frac{\partial}{\partial t} \psi_t(x) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi_t(x) + V(x)\psi_t(x)}$$

Hamiltonian $H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + M_{V(x)}$

$M_{f(x)}\, g = f(x)g(x)$ (mult by f)

Sch eq:

$$\boxed{i\hbar\frac{\partial}{\partial t}\psi_t = H\psi_t}$$

assume WLOG.
ψ_t is normalized
i.e. $\|\psi_t\| = (\langle\psi_t,\psi_t\rangle)^{1/2} = 1$

Explicit Solution of $i\hbar\dot{\psi}_t = B\psi_t$ where B is self adjoint

is

$\psi_t = U_t\psi_0$ where $\psi_0(x)$ = initial state ($t=0$)

where $t \mapsto U_t$ is a unitary gp:

U_t is unitary ($U_t^* = U_t^{-1}$)

$U_{t+s} = U_t U_s$

explicitly $U_t = e^{-iBt/\hbar}$

$B = i\hbar \frac{d}{dt}U_t\Big|_{t=0}$

i.e. B is the (Lie Alg) gen of the 1 parameter (Lie) Group $t \mapsto U_t$

Explicitly

$$\psi_t(x) = \underbrace{e^{-iHt/\hbar}}_{U_t}\psi_0(x) = \sum_{n=0}^{\infty}\left[\frac{(-iHt)^n}{n!}\psi_0(x)\right]$$

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} \qquad = \sum_{n=0}^{\infty}\frac{(-it)^n}{n!}H^n\psi_0(x)$$

$$= \sum_{n=0}^{\infty} \left[\frac{(-it)^n}{n!} \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + M_{V(x)} \right)^n \psi_0(x) \right]$$

Spectral theorem

If B is self-adjoint

then $L^2(\mathbb{R}) = \bigoplus_{n=1}^{\infty} S_n$ — S_n: eigenspace corresp. to λ_n; $S_n \perp S_m$ if $n \neq m$

where $Bf = \lambda_n f$ if $f \in S_n$

λ_n ← nth eigenvalue

i.e. $B = \sum_{n=0}^{\infty} \lambda_n P_{S_n}$ — P_{S_n} = ortho proj. onto S_n

continuous: $B = \int \lambda \, dE_\lambda$

$P_{S_n} P_{S_m} = 0$ if $n \neq m$

$P_{S_n}^2 = P_{S_n}$ — P_n self-adjoint

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thus if we write $\boxed{\psi_0(x) = \sum_{n=0}^{\infty} a_n e_n}$

$e_n \in S_n$ — e_n on basis for $L^2(\mathbb{R})$

$He_n = \lambda_n e_n$

$$H\psi_0 = \sum_{n=0}^{\infty} a_n \lambda_n e_n$$

So

$$\psi_t(x) = U_t \psi_0 = e^{-iHt/\hbar} \psi_0 = \boxed{\sum_{n=0}^{\infty} a_n e^{-i\lambda_n t/\hbar} e_n(x)}$$

Born Rule : Observable A $[A, \frac{\partial}{\partial t}] = 0$

$$\overline{A}_t = \langle A\psi_t, \psi_t \rangle = \int (A\psi_t)\overline{\psi_t}\, dx$$

prod rule

$$\frac{d}{dt}\overline{A}_t = \langle \frac{\partial}{\partial t} A\psi_t, \psi_t \rangle + \langle A\psi_t, \frac{d}{dt}\psi_t \rangle$$

$$= \langle A \frac{\partial}{\partial t}\psi_t, \psi_t \rangle + \langle A\psi_t, \frac{\partial}{\partial t}\psi_t \rangle$$

$$= \langle A \frac{1}{i\hbar} H\psi_t, \psi_t \rangle + \langle A\psi_t, \frac{1}{i\hbar} H\psi_t \rangle$$

$$= \frac{1}{i\hbar}\left[\langle AH\psi_t, \psi_t \rangle - \langle A\psi_t, H\psi_t \rangle \right]$$

$$= \frac{1}{i\hbar}\left[\langle (AH - HA)\psi_t, \psi_t \rangle \right]$$

$$= \frac{1}{i\hbar} \langle [A, H]\psi_t, \psi_t \rangle$$

so $$\boxed{\frac{d}{dt}\overline{A}_t = \frac{1}{i\hbar}\overline{[A, H]}}$$

op version of Nöether's Thm

if A is the inf gen of a symmetry $[A, H] = 0$

so $\frac{d}{dt}\overline{A}_t = 0$ const of motion

Uncertainty Principle

self adjoint ops A, B

define variance of observable T

$$\sigma_T^2 = E((T - \overline{T})^2) = \langle (T - \overline{T})\psi_t, (T - \overline{T})\psi_t \rangle$$

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2}\, \overline{\frac{1}{i}[A, B]}\right)^2$$

T self-adj
$\overline{T} = \langle T\psi, \psi\rangle$

$$\sigma_A \sigma_B \geq \frac{1}{2} \left| \overline{\frac{1}{i}[A, B]} \right|$$

Let's look at self-adj op M_x

$$\left(\overline{M_x}\right)_t = \langle M_x \psi_t, \psi_t\rangle = \int_{-\infty}^{\infty} x \psi_t(x) \overline{\psi_t(x)}\, dx$$

$$= \int_{-\infty}^{\infty} x |\psi_t(x)|^2 dx$$

Now assume $\psi_t(x)$ is normalized i.e.
$|\psi_t|^2$ is a prob density

$$= \int_{-\infty}^{\infty} x \underbrace{|\psi_t(x)|^2 dx}_{\text{prob measure}} = \breve{E}(x)_t$$

i.e. $\overline{M_x} = \breve{E}(x)$

the observable M_x can be interpreted as position

i.e. x "tracks" position, in a quantum sense

i.e. Born rule gives connection between x and classical position

velocity

$$\frac{d}{dt} \overline{x}_t = \frac{d}{dt}\left(\overline{M_x}\right)_t = \overline{\frac{1}{i\hbar}[M_x, H]}$$

$$[M_x, H]f = x\left(\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}f + \cancel{V(x)f(x)}\right) - \left(\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \cancel{V(x)}\right)(xf(x))$$

$$= -\frac{\hbar^2}{2m}\left(xf'' - (xf)''\right)$$

$$= -\frac{\hbar^2}{2m}\left(xf'' - \frac{d}{dx}(xf' + f)\right)$$

$$= -\frac{\hbar^2}{2m}\left(\cancel{xf''} - \cancel{xf''} - f' - f'\right)$$

$$= \frac{\hbar^2}{m}f' = \left(\frac{\hbar^2}{m}\frac{\partial}{\partial x}\right)f \Rightarrow [M_x, H] = \frac{\hbar^2}{m}\frac{\partial}{\partial x}$$

so $$\frac{d}{dx}\bar{x}_t = \overline{\frac{1}{i\hbar}\frac{\hbar^2}{m}\frac{\partial}{\partial x}}$$

$$\underbrace{m\frac{d}{dt}\bar{x}_t}_{} = \overline{\frac{\hbar}{i}\frac{\partial}{\partial x}}$$

ave momentum $\bar{P}_t = \overline{\frac{\hbar}{i}\frac{\partial}{\partial x}}$ So Born Rule $\Rightarrow$ $\frac{\hbar}{i}\frac{\partial}{\partial x} = P$ can be interpreted as Momentum

uncertainty $\sigma_x \sigma_p \geq \frac{\hbar}{2}$

Next time

Sch Eq in Momentum space

$$i\hbar\frac{\partial}{\partial t}\varphi_t(p) = \frac{P^2}{2m}\varphi_t(p) + \hat{V}_h * \varphi_t(p)$$

$$V_h(x) = V(hx)$$

$$\varphi_t = \frac{1}{\sqrt{h}}\hat{\psi}_t\left(\frac{p}{h}\right)$$

$$i\hbar \frac{\partial \psi_t}{\partial t} = \underbrace{-\frac{\hbar^2}{2m} \frac{\partial^2 \psi_t}{\partial x^2}} + V(x)\psi(x)$$

$$= \frac{P^2}{2m}\psi_t \qquad P^2 = \left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2 = \frac{\hbar^2}{-1}\frac{\partial^2}{\partial x^2} = -\hbar^2\frac{\partial^2}{\partial x^2}$$

$\downarrow$ transformation

$$\psi_t(x) \to \varphi_t(p)$$

$$\frac{P^2}{2m} \longleftrightarrow \frac{M_p^2}{2m}$$

$$\left.\begin{array}{l} p \longleftrightarrow M_p \\ \frac{\hbar}{i}\frac{\partial}{\partial x} \longleftrightarrow M_p \\ \text{diff} \longleftrightarrow \text{mult} \end{array}\right\} \text{can think of real \# } p \text{ as being "momentum" variable}$$

$$\underbrace{\text{diff} \longleftrightarrow \text{mult}}_{\text{mult be Fourier transf.}}$$

Fourier Transf

$$\mathcal{F}(f)(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x\xi} dx = \hat{f}(\xi)$$

$\mathcal{F}$ is unitary: 1-1, onto, preserves inner products (thus lengths)

$$\mathcal{F}(f')(\xi) = \int_{-\infty}^{\infty} f'(x) e^{-2\pi i \xi x} dx \qquad f \to 0 \text{ at } \infty$$

$$\underset{\text{parts}}{=} -\int_{-\infty}^{\infty} f(x)(-2\pi i\xi) e^{-2\pi i x\xi} dx$$

$$\widehat{(f')}(\xi) = 2\pi i\xi \hat{f}(\xi)$$

$$\mathcal{F}\left(\frac{\partial}{\partial x}\right) = M_{2\pi i\xi}\mathcal{F} \qquad \mathcal{F} \text{ intertwines } \frac{\partial}{\partial x} \text{ \& } M_{2\pi i\xi}$$

$$\mathcal{F}\left(\frac{1}{i}\frac{\partial}{\partial x}\right) = M_{2\pi\xi}\,\mathcal{F}$$

want

$$\text{"}\mathcal{F}\text{"}\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right) = M_\xi\, \text{"}\mathcal{F}\text{"}$$

define $${}_h\mathcal{F} f(\xi) = \frac{1}{\sqrt{h}} \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x\xi/h}\, dx$$

$$= \frac{1}{\sqrt{h}}\, \mathcal{F}(f)\left(\frac{\xi}{h}\right)$$

scaled fourier's transform

Define $$\varphi_t(\xi) = {}_h\mathcal{F}(\psi_t)(\xi)$$

Sch Eq

$${}_h\mathcal{F}\left(i\hbar \frac{\partial \psi_t(x)}{\partial t}\right) = {}_h\mathcal{F}\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi_t(x) + V(x)\,\psi_t(x)\right)$$

$$i\hbar \frac{\partial \varphi_t(\xi)}{\partial t} = \frac{\xi^2}{2m}\varphi_t(\xi) + V_h * \varphi_t(\xi)$$

$$V_h(x) = V(hx)$$

$$f * g(\xi) = \int_{-\infty}^{\infty} f(t)\, g(\xi - t)\, dt$$

To argue $\overline{P} = \overline{\dfrac{\xi^2}{2m}}$ $\quad |\varphi_t|^2$ is a prob dist

r_0 { parameterizes momentum

$$i\hbar \frac{\partial \psi_t}{\partial t}(x) = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi_t}{\partial x^2} + V(x)\psi_t(x)$$

$$\psi_t(x) = e^{-iHt/\hbar}\psi_0(x) \qquad H = -\frac{\hbar^2}{2m}\partial_x^2 + M_{V(x)}$$

$$\mathcal{J}_h f(\xi) = \frac{1}{\sqrt{h}}\hat{f}\left(\frac{\xi}{h}\right)$$

$\mathcal{J}_h : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is unitary

preserves $\langle\ ,\ \rangle$

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx$$

$$f \in L^2 \qquad f: \mathbb{R} \to \mathbb{C}$$

$$\psi_t(x) \longrightarrow \varphi_t(\xi) = \mathcal{J}_h \psi_t(\xi) = \frac{1}{\sqrt{h}}\hat{\psi}_t\left(\frac{\xi}{h}\right)$$

$$\mathcal{J}_h\left(i\hbar\frac{\partial \psi_t}{\partial t}\right) = \mathcal{J}_h\left(H\psi_t\right)$$

New Sch Eq

$$\boxed{i\hbar\,\varphi_t(\xi) = \frac{\xi^2}{2m}\varphi_t(\xi) + \left(\hat{V}_h * \varphi_t\right)(\xi)}$$

$$V_h(x) = V(hx)$$

$$f*g(\xi) = \int_{-\infty}^{\infty} f(t)\,g(\xi - t)\,dt \qquad (fg)^\wedge = \hat{f}*\hat{g} \qquad (f*g)^\wedge = \hat{f}\hat{g}$$

Born rule:

$$\langle P\rangle_t = \langle P\psi_t, \psi_t\rangle = \left\langle \frac{\hbar}{i}\frac{\partial}{\partial x}\psi_t, \psi_t\right\rangle \qquad P \longleftrightarrow \frac{\hbar}{i}\frac{\partial}{\partial x}$$

$$= \left\langle \underset{h}{\mathcal{F}} \frac{\hbar}{i}\frac{\partial}{\partial x}\psi_t, \underset{h}{\mathcal{F}} \psi_t \right\rangle \qquad \langle f, g\rangle = \langle \underset{h}{\mathcal{F}} f, \underset{h}{\mathcal{F}} g\rangle$$

$$= \langle \xi \varphi_t , \varphi_t \rangle$$

why?

$$\underset{h}{\mathcal{F}}\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\psi_t\right)(\xi) = \frac{1}{\sqrt{h}}\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\psi_t\right)^{\wedge}\left(\frac{\xi}{h}\right)$$

Parts: $\int_a^b f g' = fg\Big|_a^b - \int_a^b f' g$

$$= \frac{1}{\sqrt{h}}\frac{\hbar}{i}\int_{-\infty}^{\infty} dx\, e^{-2\pi i x\left(\frac{\xi}{h}\right)} \frac{\partial}{\partial x}\psi_t(x) \qquad \left(\psi_t\Big|_{-\infty}^{\infty} = 0\right)$$

(parts)

$$= -\frac{1}{\sqrt{h}}\frac{\hbar}{i}\int_{-\infty}^{\infty} dx \left(-2\pi i \frac{\xi}{h}\right) e^{-2\pi i x\left(\frac{\xi}{h}\right)} \psi_t$$

$$= -\frac{1}{\sqrt{h}}\frac{\hbar}{i}\left(-2\pi i \frac{\xi}{h}\right)\hat{\psi}_t\left(\frac{\xi}{h}\right)$$

$$= \xi \frac{1}{\sqrt{h}}\hat{\psi}_t\left(\frac{\xi}{h}\right) = \xi \varphi_t(\xi)$$

$$\langle P\rangle_t = \langle \xi\varphi_t, \varphi_t\rangle = \int_{-\infty}^{\infty} \xi \underbrace{|\varphi_t(\xi)|^2}_{\text{prob distribution}} d\xi$$

$$= \langle \xi \rangle_t$$

therefore $\xi \leftrightarrow$ momentum

$$\boxed{\xi = p}$$

So ξ space is momentum space

thus φ is wave function in momentum space

and in momentum space

$$\boxed{i\hbar \frac{\partial \varphi_t(p)}{\partial t} = H\varphi_t(p)}$$

$$H = M_{p^2/2m} + \hat{V}_\hbar * (\ )$$

$$H\varphi_t(p) = \frac{p^2}{2m}\varphi_t(p) + (\hat{V}_\hbar * \varphi_t)(p)$$

In free space $V \equiv 0$

So

$$i\hbar \frac{\partial \varphi_t}{\partial t}(p) = \frac{p^2}{2m}\varphi_t(p) \qquad H = M_{p^2/2m}$$

$$\varphi_t(p) = U_t\varphi_0(p) = e^{-iHt/\hbar}\varphi_0(p)$$

$$= e^{-iM_{p^2/2m}t/\hbar}\varphi_0(p)$$

$$\boxed{\varphi_t(p) = e^{-ip^2t/2m\hbar}\varphi_0(p)}$$

why?

$$e^{-iHt/\hbar} f = \sum_{n=0}^{\infty}\left[\frac{1}{n!}\left(\frac{-iHt}{\hbar}\right)^n f\right]$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-ip^2t}{2m\hbar}\right)^n f(p)$$

$$= \left[\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-ip^2t}{2m\hbar}\right)^n\right] f(p)$$

$$= e^{-ip^2t/2m\hbar} f(p)$$

$$\langle P\rangle_t = \int_{-\infty}^{\infty} p|\varphi_t|^2\,dp = \int_{-\infty}^{\infty} p|\varphi_0|^2\,dp = \text{const}!$$

Gaussian Wave packet in free space

$$\psi_0(x) = \left(\frac{1}{\sqrt{2\pi}\sigma_0}\right)^{1/2} e^{-x^2/4\sigma_0} \qquad \|\psi_0\| = 1$$

$$\varphi_0(p) = \frac{1}{\sqrt{\hbar}}\hat{\psi}_0\left(\frac{p}{\hbar}\right) = \left(\frac{1}{\sqrt{2\pi}\sigma_p}\right)^{1/2} e^{-p^2/4\sigma_p^2} \qquad \sigma_p^2 = \frac{\hbar^2}{4\sigma_0^2}$$

$$\sigma_p^2\sigma_x^2 = \frac{\hbar^2}{4\sigma_0^2}\sigma_0^2 = \frac{\hbar^2}{4}$$

$$\sigma_p\sigma_x = \frac{\hbar}{2}$$

min of uncert principle

$$\varphi_t(p) = e^{-ip^2t/2m\hbar}\varphi_0(p)$$

$$\psi_t(x) = \mathcal{F}_\hbar^{-1}\varphi_t(x) = \frac{1}{\sqrt{\hbar}}\check{\varphi}_t\left(\frac{x}{\hbar}\right)$$

$$\boxed{\psi_t(x) = \left(\frac{\sigma_0}{\sqrt{2\pi}\sigma_t^2}\right)^{1/2} e^{-x^2/4\sigma_t^2}}$$

$$\psi_t(x) = |\psi_t(x)|e^{i\theta_t}$$

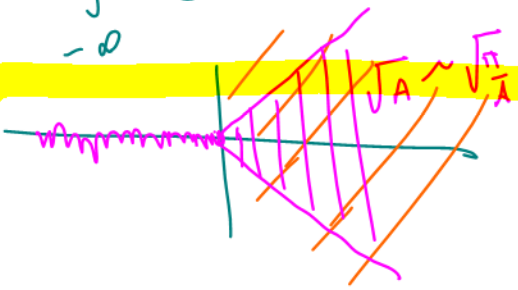
$|\psi_t(x)|^2$

$$\int|\psi_t|^2dx = 1$$

$$\sigma_t^2 = \sigma_0^2 + \frac{i\hbar t}{2m}$$

A, B complex, $\mathrm{Re}\,A > 0$

Hinges on:
$$\boxed{\int_{-\infty}^{\infty} e^{-A(x+B)^2}dx = \sqrt{\frac{\pi}{A}} \quad (\mathrm{Re}\,A > 0)}$$



Prob density

$$|\psi_t|^2 = \psi_t\overline{\psi_t}$$

$$|Ae^B|^2 = |A|^2 \underbrace{e^B\overline{e^B}}$$

$$|e^{5+6i}|^2 = |e^5e^{6i}|^2 = |e^5|^2|e^{6i}|^2 = e^{10}$$

$$e^Be^{\bar{B}} = e^{2\,\mathrm{Re}\,B}$$

$$|\psi_t|^2 = \left| \left(\frac{\sigma_0}{\sqrt{2\pi}\,\sigma_t^2}\right)^{1/2} e^{-x^2/4\sigma_t^2} \right|^2$$

$$\frac{1}{\sqrt{2\pi}\,\sigma} e^{-x^2/2\sigma^2}$$

$$= \underbrace{\left| \frac{\sigma_0}{\sqrt{2\pi}\,\sigma_t^2} \right|}_{\frac{1}{\sqrt{2\pi}\,\Sigma_t}} e^{-2\mathrm{Re}\,\frac{x^2}{4\sigma_t^2}}$$

$$\boxed{\sigma_t^2 = \sigma_0^2 + \frac{i\hbar t}{2m}}$$

$$|\sigma_t|^2 = \left(\sigma_0^4 + \frac{\hbar^2 t^2}{4m^2}\right)^{1/2}$$

$$|a+ib| = \sqrt{a^2+b^2}$$

$$\frac{\sigma_0}{\sqrt{2\pi}\,|\sigma_t^2|} = \frac{\sigma_0}{2\pi} \frac{1}{\left|\sigma_0^2 + \frac{i\hbar t}{2m}\right|}$$

$$= \frac{\sigma_0}{\sqrt{2\pi}} \frac{1}{\left(\sigma_0^4 + \frac{\hbar^2 t^2}{4m^2}\right)^{1/2}}$$

$$t=0 \quad = \frac{1}{\sqrt{2\pi}\,\sigma_0}$$

want

$$= \frac{1}{\sqrt{2\pi}\,\Sigma_t}$$

$$\Sigma_t = \frac{|\sigma_t|^2}{\sigma_0}$$

$$\Sigma_t \sigma_0 = |\sigma_t|^2$$

$$\Sigma_t = \frac{\left(\sigma_0^4 + \frac{\hbar^2 t^2}{4m^2}\right)^{1/2}}{\sigma_0} = \left(\sigma_0^2 + \frac{\hbar^2 t^2}{4m^2 \sigma_0^2}\right)^{1/2}$$

$$e^{-2\mathrm{Re}\,\frac{x^2}{4\sigma_t^2}} = e^{-\frac{x^2}{2}\mathrm{Re}\,\frac{1}{\sigma_t^2}}$$

$$\mathrm{Re}\,\frac{1}{\sigma_t^2} = \mathrm{Re}\,\frac{1}{\sigma_0^2 + \frac{i\hbar t}{2m}}$$

$$\mathrm{Re}\,\frac{1}{a+bi}$$

$$= \mathrm{Re}\,\frac{a-bi}{|a+bi|^2}$$

$$= \frac{\sigma_0^2}{\sigma_0^4 + \frac{\hbar^2 t^2}{4m^2}} = 1/\Sigma_t^2$$

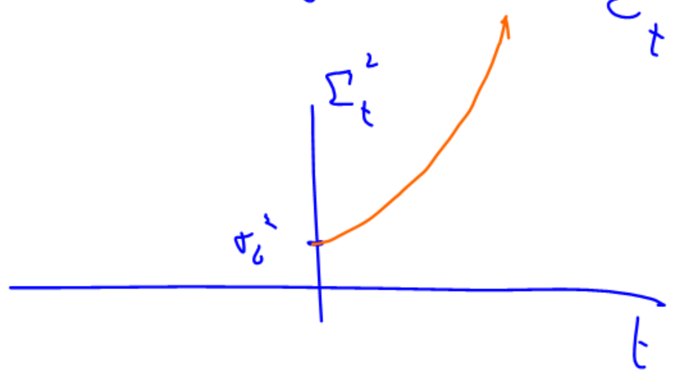
$= \frac{a}{a^2+b^2}$

Summary

$$|\psi_t(x)|^2 = \frac{1}{\sqrt{2\pi}\,\Sigma_t} e^{-x^2/2\Sigma_t^2}$$

= gaussian prob dist w/

variance $\Sigma_t^2 = \dfrac{\sigma_0^4 + \frac{\hbar^2 t^2}{4m^2}}{\sigma_0^2}$



$$\Sigma_t = \sqrt{\sigma_0^2 + \frac{\hbar^2 t^2}{4m^2\sigma_0^2}}$$

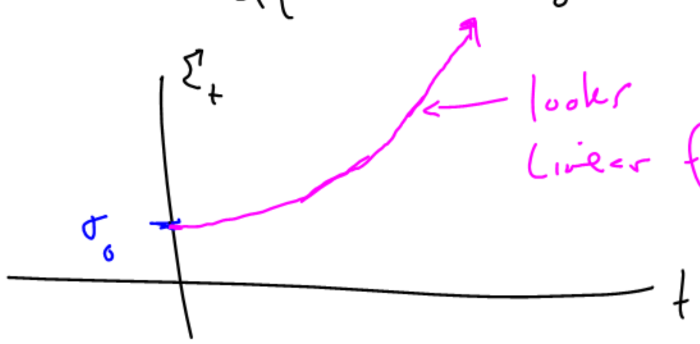
$$= \frac{\hbar t}{2m\sigma_0}\sqrt{\frac{4m^2\sigma_0^4}{\hbar^2 t^2} + 1} \rightarrow \frac{\hbar t}{2m\sigma_0}$$

0 as $t \to \infty$

for large t

$\hbar = v \cdot x \cdot m$

So $\dfrac{d\Sigma_t}{dt} \rightarrow \dfrac{\hbar}{2m\sigma_0}$ for large t



$f(s) = \sqrt{a + bs^2} \quad a, b > 0$

$$f'(r) = \frac{br}{\sqrt{a+br^2}}$$

$$f''(r) = \frac{\sqrt{a+br^2}\,(b) - br\,\frac{br}{\sqrt{a+br^2}}}{a+br^2}$$

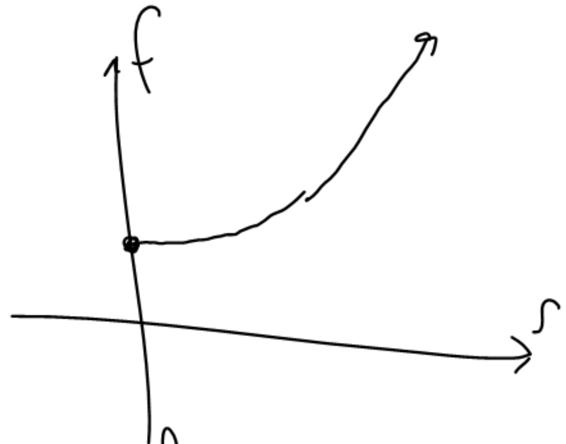
$\sqrt{a+br^2}$ top $= (a+br^2)b - br\,br$

$= ab + b^2r^2 - b^2r^2$

$= ab + \cancel{(b^2-b^2)r^2}$

> 0 boom! $f'' > 0$ $f' = 0$ when $r = 0$

$f' > 0$ when $r > 0$



<u>velocity of spread</u>

$$\frac{d\Sigma_t}{dt} \begin{cases} \approx 0 & \text{when } t \text{ small} \\ \to \frac{h}{2m r_0} & \text{when } t \text{ grows} \end{cases}$$

v_{max} = max velocity of spread $\approx \frac{h}{2m r_0}$

<u>Hydrogen</u> $r_0 \approx 10^{-10}$ m

$m \approx 9.11 \times 10^{-31}$ kg $h \approx 6.63 \times 10^{-34}$

$v_{max} \approx 6 \times 10^5$ m/s $\approx \frac{1}{500}$ speed of light

Uncertainty Princip

Min uncertainty

$$\Delta x \underbrace{\Delta p}_{m \Delta v} = \frac{\hbar}{2}$$

$$\Delta v = \frac{\hbar}{2m \Delta x} = \frac{\hbar}{2m \sigma_0} \quad \text{as before}$$