Clifford algebra and Wick rotation notes

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Contents

These notes make use of

- José Figueroa-o'Farill's online notes (Majorana.pdf) on Clifford algebras and their representations
- Fulton and Harris – Representation Theory: A First Course
- Jean Gallier's online notes (clifford.pdf) on Clifford algebras
- The wikipedia page on higher-dimensional gamma matrices
- The lecture notes on Clifford algebras by Lundholm and Svensson.
- The wikipedia page on the spin group
- Varadarajan - Supersymmetry for mathematicians
- Many posts on math stackexchange and math overflow

and other resources. There are some errors or bad exposition in some of the above resources, and (to the best of my ability) there are rewrites/corrections here. We focus mainly on the case of real/complex Clifford algebras (as we seek to characterise Wick rotation eventually), but a decent amount of techniques will readily extend to fields other than $\mathbb{R}, \mathbb{C}$.

Note: Every vector space/algebra is assumed to be finite-dimensional. There are quite a few times where I leverage finite-dimensionality without explicitly stating it, typically in the use of a finite basis. Perhaps the most subtle manifestation of this is done in proving Theorem 3.3.

1 Structure of real Clifford algebras $\mathrm{Cl}(p, q)$

1.1 Definitions and conventions

Let $V = \mathbb{R}^{p,q}$ be endowed with the metric $\eta_{ab} = \mathrm{diag}(1, \dots, -1)$. We will take the Clifford algebra $\mathrm{Cl}(p, q)$ to be generated by products of Γ_a obeying

$$\Gamma_a\Gamma_b + \Gamma_b\Gamma_a = -2\eta_{ab}\,\mathrm{id}\ . \tag{1.1}$$

If one wants to identify this with the construction by quotienting the tensor algebra of $\mathbb{R}^{p,q}$, then Γ_a correspond to the usual basis vectors e_a of $\mathbb{R}^{p,q}$ such that $\eta(e_a, e_b) = \eta_{ab}$ and one writes instead

$$e_ae_b + e_be_a = -2\eta_{ab}\,\mathrm{id}\ . \tag{1.2}$$

This formulation also reveals the natural inclusion of the underlying vector space $\mathbb{R}^{p,q} \to \mathrm{Cl}(p, q)$. A basis of $\mathrm{Cl}(p, q)$ is then given by the 2^{p+q} vectors $\{\mathrm{id}, e_{a_1} \dots e_{a_n}\}$ where $a_1 < \dots < a_n$ and $n = 1, \dots, p + q$, very analogous to the exterior algebra $\Lambda(\mathbb{R}^{p,q})$.

We briefly mention that the complex Clifford algebra $\mathbb{C}\mathrm{l}(d)$ of $\mathbb{C}^d$ endowed with the Euclidean bilinear form $\delta = \mathrm{diag}(1, \dots, 1)$ is isomorphic to the complexification $\mathrm{Cl}(p, q)$ of any real Clifford algebra with $d = p + q$. This is obvious through a change of basis using i so that η becomes δ.

1.2 Natural maps

We now look at some natural linear maps $\Phi : \mathrm{Cl}(p, q) \to \mathrm{Cl}(p, q)$ which preserve the multiplication defined by (1.2). The 'natural' part means that it is independent of the basis we originally choose for $V \cong \mathbb{R}^{p,q}$. Hence we will work explicitly in a basis-independent setting with $\mathrm{Cl}(V, \eta)$

defined by its construction as a quotient of the tensor algebra. Here we just have a natural inclusion map $V \to \mathrm{Cl}(V,\eta)$ and the relation

$$vw + wv = -2\eta(v,w) \tag{1.3}$$

for some symmetric non-degenerate bilinear form η with signature (p,q). Such maps Φ then obey

$$\Phi(vw + wv) = vw + wv = -2\eta(v,w)\,, \tag{1.4}$$

where we additionally require that Φ restricts to a linear map $V \to V$. Sometimes these maps are entirely determined by their restriction $V \to V$ and the rule $\Phi(vw) = \Phi(v)\Phi(w)$. In this case the condition (1.4) can be rewritten as $\Phi \in O(V,\eta)$.

1.2.1 Signature-independent maps

Two important examples are the *main involution* α and the *reversal* or *main anti-automorphism* τ defined by

$$\alpha(v) := -v\,, \tag{1.5}$$
$$\tau(v_1 \dots v_n) := v_n \dots v_1\,, \tag{1.6}$$

where $v, v_1, \dots, v_n \in V$ are arbitrary vectors. Indeed, we can see that

$$\alpha(vw + wv) = \alpha(v)\alpha(w) + \alpha(w)\alpha(v) = (-1)^2(vw + wv)\,, \tag{1.7}$$
$$\tau(vw + wv) = \tau(vw) + \tau(wv) \qquad = wv + vw\,, \tag{1.8}$$

both preserve Clifford multiplication. Note that $\tau(xy) = \tau(y)\tau(x)$ in general; this is why it is called an *anti*-automorphism.

Both maps α and τ are independent of the signature (p,q). In fact, $\alpha : V \to V$ which sends $v \mapsto -v$ is just an orthogonal linear map on V which we have lifted to the Clifford algebra. Physically, $\alpha = PT \in O(V,\eta)$ which flips both space and time directions.

1.2.2 A signature-dependent map

An involution β which depends on the signature (p,q) can be constructed starting from a map $\beta : V \to V$ which depends on the signature. Indeed, we just want to choose $\beta = P$ or $\beta = T$. The interpretation depends on a choice of what part of V is temporal and what part of V is spatial, i.e. a choice of decomposition $V = V_1 \oplus V_2$ where $V_1 \perp V_2$ are orthogonal and $\eta|_{V_1}$ is positive-definite and $\eta|_{V_2}$ is negative-definite.[1] With the choice made, define

$$\beta(v_1) := v_1\,, \tag{1.9}$$
$$\beta(v_2) := -v_2\,. \tag{1.10}$$

Sadly, such a map is not natural (basis-independent) because we had to choose a decomposition $V = V_1 \oplus V_2$, of which there are infinitely many, even in the case $\dim V = 2$, $(p,q) = (1,1)$. However, it is independent of a choice of basis for V_1 and for V_2. Hence one could at least say it is natural for $\mathrm{Cl}(V_1 \oplus V_2, \eta)$.

We can also readily define $\gamma = \alpha\beta = \beta\alpha$ which acts as T if β acts as P. Explicitly,

$$\gamma(v_1) := -v_1\,, \tag{1.11}$$
$$\gamma(v_2) := v_2\,. \tag{1.12}$$

These maps β, γ essentially correspond to the complex transpose of a given matrix rep of the Clifford algebra. We will make this more precise later and relate it to pinor and spinor reps.

[1] We take V_1 to be temporal, so $\beta = P$.

1.2.3 Clifford algebras are superalgebras

A useful point is that $\alpha(v) = -v$ clearly decomposes $\mathrm{Cl}(V,\eta)$ into two parts

$$x = \frac{1}{2}(x + \alpha(x)) + \frac{1}{2}(x - \alpha(x)) , \tag{1.13}$$

$$\mathrm{Cl}(V,\eta) = \mathrm{Cl}(V,\eta)^0 \oplus \mathrm{Cl}(V,\eta)^1 , \tag{1.14}$$

where $\mathrm{Cl}(V,\eta)^0$ contains linear combinations of even products ($\alpha(x) = x$) and $\mathrm{Cl}(V,\eta)^1$ contains linear combinations of odd products ($\alpha(x) = -x$).

It is obvious that Clifford multiplication preserves this $\mathbb{Z}_2$-grading, i.e.

$$\mathrm{Cl}(V,\eta)^i \oplus \mathrm{Cl}(V,\eta)^j \subseteq \mathrm{Cl}(V,\eta)^{i+j} , \tag{1.15}$$

where $i + j$ is interpreted mod 2. In particular, the product of two evens is even, the product of two odds is even and the product of an even and an odd is odd.

An interesting point is that any subset of $\mathrm{Cl}(V,\eta)$ will naturally inherit these grading properties, including the group of units $\mathrm{Cl}(V,\eta)^\times$. One could call it a 'supergroup', but this is bad terminology as it typically means a super Lie group, i.e. a group in the category of supermanifolds.

1.3 Lipschitz–Clifford group and pin and spin groups

We first consider how to realise reflections in a Clifford algebraic way. Then develop the tools needed to define *pin* and *spin groups* as groups living inside $\mathrm{Cl}(V,\eta)$.

The idea is that we will first establish a group $\Gamma(V,\eta) \subseteq \mathrm{Cl}(V,\eta)$ called the *Lipschitz–Clifford group* or *Lipshitz group* or *Clifford group* which has a rep on V

$$\widetilde{\mathrm{Ad}} : \Gamma(V,\eta) \to GL(V) \tag{1.16}$$

such that its image contains all possible compositions of reflections and, by Cartan–Dieudonné theorem, the entire orthogonal group, i.e. $O(V,\eta) \subseteq \mathrm{im}\,\widetilde{\mathrm{Ad}}$ (in fact $O(V,\eta) = \mathrm{im}\,\widetilde{\mathrm{Ad}}$). We will then find a method to identify the appropriate subgroup $\mathrm{Pin}(V,\eta) \subseteq \Gamma(V,\eta)$ corresponding to the pin group, at which point we will have the rep

$$\widetilde{\mathrm{Ad}} : \mathrm{Pin}(V,\eta) \to O(V,\eta) \tag{1.17}$$

which is also a covering map with $\mathrm{Pin}(V,\eta)$ the double (and, depending on (p,q), universal) cover of $O(V,\eta)$.

This will readily descend to the case of the even subgroup $\mathrm{Spin}(V,\eta) := \mathrm{Pin}^0(V,\eta) := \mathrm{Pin}(V,\eta) \cap \mathrm{Cl}(V,\eta)^0$ with

$$\widetilde{\mathrm{Ad}} : \mathrm{Spin}(V,\eta) \to SO(V,\eta) . \tag{1.18}$$

We will also define the orthochronous spin group $\mathrm{Spin}^+(V,\eta) \subseteq \mathrm{Spin}(V,\eta)$ which will always be connected, except for $(p,q) = (1,0),(0,1),(1,1)$. Equality $\mathrm{Spin}^+(V,\eta) = \mathrm{Spin}(V,\eta)$ is only obtained for definite forms η with $(p,q) = (0,d),(d,0)$.

We will also give a more explicit presentation of pin and spin as groups of the form

$$\mathrm{Pin}(V,\eta) = \{v_1 \dots v_r \mid \eta(v_i) = \pm 1\} \subseteq \mathrm{Cl}(p,q) , \tag{1.19}$$

$$\mathrm{Spin}(V,\eta) = \{v_1 \dots v_{2r} \mid \eta(v_i) = \pm 1\} = \mathrm{Pin}(p,q) \cap \mathrm{Cl}(p,q)^0 , \tag{1.20}$$

where $\eta(v) := \eta(v,v)$. That is, each element is merely a product of vectors with non-zero norms.

1.3.1 Reflections

Consider some basis-free formulation (V, η). Given some $v \in V$ such that $\eta(v, v) \neq 0$, we can reflect about the hyperplane orthogonal to v by

$$\sigma_v(w) = w - 2\frac{\eta(v, w)}{\eta(v, v)}v\,. \tag{1.21}$$

Geometrically, one wants to translate w far enough in the direction v so that their 'dot product' changes signs, i.e. $\eta(\sigma_v(w), v) = -\eta(w, v)$. The following lemma gives their main properties.

Lemma 1.1. *Let $\sigma_v : V \to V$ be a reflection, then σ_v is linear, $\sigma_v \in O(V, \eta)$ and $\det(\sigma_v) = -1$.*

Proof. Reflections are obviously linear by virtue of their formula. Checking $\sigma_v \in O(V, \eta)$ just amounts to computing $\eta(\sigma_v(w), \sigma_v(u))$, so we skip this.

Checking $\det(\sigma_v) = -1$ is more interesting. Notice v determines an orthogonal decomposition $V = \mathrm{span}(v) \oplus H$. Moreover, it is clear that $\sigma_v(v) = -v$ and $\sigma_v(h) = h$. Hence, in any basis of H, σ_v has only eigenvalue 1 on each basis vector, whilst it has eigenvalue -1 on any basis of $\mathrm{span}(v)$. Thus, $\det(\sigma_v) = \prod \lambda_i = -1$. □

Finally, a key theorem is that reflections actually generate the orthogonal group!

Theorem 1.2 (Cartan–Dieudonné)**.** *Let (V, η) be a $d = p + q$-dimensional vector space over a field k with $\mathrm{char}(k) \neq 2$ and with non-degenerate symmetric bilinear form η. Then every element of $O(V, \eta)$ is a composition of at most d reflections.*

Similarly, even products of reflections must generate the special orthogonal group.

1.3.2 The twisted adjoint representation and Clifford-Lipschitz group

If we consider reflections in context of the Clifford algebra by leveraging the natural inclusion $V \to \mathrm{Cl}(V, \eta)$, then one may write

$$\begin{aligned}
\sigma_v(w) &= w - \frac{2\eta(v, w)}{\eta(v, v)}v && (1.22)\\
&= w - \frac{-(vw + wv)}{(-v^2)}v && (1.23)\\
&= w - \frac{(vwv + wv^2)}{v^2} && (1.24)\\
&= -vwv^{-1} && (1.25)\\
&= \alpha(v)wv^{-1} && (1.26)\\
&=: \widetilde{\mathrm{Ad}}(v)w\,. && (1.27)
\end{aligned}$$

Thus, reflections $\sigma_v : V \to V$ are realised naturally by the Clifford algebra by *twisted conjugation* or *twisted adjoint representation.*

This *twisted adjoint representation* is quite an interesting concept to make more general. Let $\mathrm{Cl}(V, \eta)^\times$ denote the group of units. Let $x \in \mathrm{Cl}(V, \eta)^\times$ and $y \in \mathrm{Cl}(V, \eta)$, then we define the linear map[2]

$$\widetilde{\mathrm{Ad}}(x)y := \alpha(x)yx^{-1}\,. \tag{1.28}$$

A basic property of the twisted adjoint representation is that

$$\widetilde{\mathrm{Ad}}(x_1x_2)y = \alpha(x_1x_2)y(x_1x_2)^{-1} = \alpha(x_1)\alpha(x_2)yx_2^{-1}x_1^{-1} = \widetilde{\mathrm{Ad}}(x_1)\widetilde{\mathrm{Ad}}(x_2)y\,. \tag{1.29}$$

Consequently, $\widetilde{\mathrm{Ad}}(x)$ is invertible with inverse $\widetilde{\mathrm{Ad}}(x)^{-1} = \widetilde{\mathrm{Ad}}(x^{-1})$.

We now consider the collection of $x \in \mathrm{Cl}(V, \eta)^\times$ that act as linear maps $\widetilde{\mathrm{Ad}}(x) : V \to V$ when restricted to V.

[2]This is not quite an automorphism as $\widetilde{\mathrm{Ad}}(x)(y_1y_2)$ need not always equal $\widetilde{\mathrm{Ad}}(x)(y_1)\widetilde{\mathrm{Ad}}(x)(y_2)$.

Definition 1.3 (Lipschitz–Clifford group)**.** The *Lipschitz–Clifford group* $\Gamma(V,\eta)$ is defined as

$$\Gamma(V,\eta) := \{x \in \mathrm{Cl}(V,\eta)^\times \mid \widetilde{\mathrm{Ad}}(x)(v) \in V \quad \text{for all } v \in V\}\,. \tag{1.30}$$

Similarly, we define the *special Lipschitz–Clifford group* by $\Gamma^0(V,\eta) := \Gamma(V,\eta) \cap \mathrm{Cl}(V,\eta)^0$.

This indeed defines a group. If $x_1, x_2 \in \Gamma(V,\eta)$, then $x_1x_2 \in \Gamma(V,\eta)$ because

$$\widetilde{\mathrm{Ad}}(x_1x_2)v = \widetilde{\mathrm{Ad}}(x_1)\widetilde{\mathrm{Ad}}(x_2)v \in V\,. \tag{1.31}$$

Moreover, if $x \in \Gamma(V,\eta)$, then $x^{-1} \in \Gamma(V,\eta)$ because $\widetilde{\mathrm{Ad}}(x^{-1}) = \widetilde{\mathrm{Ad}}(x)^{-1} : V \to V$. The case of the subset $\Gamma^0(V,\eta)$ being a group follows as well since the product of evens is even.

The twisted adjoint representation is a 'surjective' representation. From the above we can see that the (special) Lipschitz–Clifford groups are afforded a rep on V by $\widetilde{\mathrm{Ad}}$. That is we have group homomorphisms

$$\widetilde{\mathrm{Ad}} : \Gamma(V,\eta) \to GL(V)\,, \tag{1.32}$$
$$\widetilde{\mathrm{Ad}} : \Gamma^0(V,\eta) \to GL(V)\,. \tag{1.33}$$

Note that arbitrary products of vectors $x = v_1 \dots v_r$ exist in $\mathrm{Cl}(V,\eta)$. Indeed, this is fine, but such an expression is not unique in general. If $r > p+q$ we can guarantee such a product will decompose into a linear combination of products of vectors with $p+q \leq r$ by picking a basis of V and using Clifford multiplication (1.2).

Such a product $x = v_1 \dots v_r$ will also be a unit iff $\eta(v_i,v_i) \neq 0$ are all non-zero. This is because $\tau(x) = \lambda x^{-1}$ for some non-zero $\lambda \in \mathbb{R}$ iff $\eta(v_i,v_i) \neq 0$ are all non-zero. Moreover, $\widetilde{\mathrm{Ad}}(x)v \in V$ because this is just a composition of many reflections. Thus, we get the following result by leveraging Cartan-Dieudonné.

Proposition 1.4. *The twisted adjoint representation*

$$\widetilde{\mathrm{Ad}} : \Gamma(V,\eta) \to GL(V) \tag{1.34}$$

is indeed a rep and $O(V,\eta) \subseteq \mathrm{im}\,\widetilde{\mathrm{Ad}}$. The analogous result holds for $\Gamma^0(V,\eta)$ and $SO(V,\eta)$.

1.3.3 The Clifford norm and properties of the Clifford–Lipschitz group

Now we introduce a useful map κ defined by $\kappa := \alpha\tau = \tau\alpha$. A similar, but less useful map is $\kappa_\gamma := \gamma\tau = \tau\gamma$ and κ_γ, by virtue of using γ, is dependent on a choice of decomposition $V = V_1 \oplus V_2$. They have the following property when acting on products of orthonormal basis vectors

$$x\kappa(x) = \pm 1\,, \tag{1.35a}$$
$$x\kappa_\gamma(x) = 1\,, \tag{1.35b}$$

with $x = e_{a_1} \dots e_{a_n}$.

Definition 1.5 (Clifford norm)**.** The *Clifford norm* $N : Cl(V,\eta) \to Cl(V,\eta)$ is defined by $N(x) := x\kappa(x)$.[3]

We need a technical lemma to progress.

Lemma 1.6. *The kernel of* $\widetilde{\mathrm{Ad}} : \Gamma(V,\eta) \to GL(V)$ *is* $\ker(\widetilde{\mathrm{Ad}}) = \mathbb{R}^\times \cdot \mathrm{id}$.

[3] We avoid using $N_\gamma(x) := x\kappa_\gamma(x)$ because it is unyieldly to work with and actually loses some important properties which N has.

Proof. Let $x \in \ker(\widetilde{\mathrm{Ad}})$, then $\alpha(x)v = vx$ for all $v \in V$ when treating things inside $\mathrm{Cl}(V, \eta)$.

Since v has fixed parity 1, decomposing $x = x_0 + x_1$ yields two equations

$$x_0 v = v x_0 \,, \tag{1.36}$$
$$-x_1 v = v x_1 \,. \tag{1.37}$$

Fix a basis e_a of V which diagonalises η as $\eta_{ab} = \mathrm{diag}(1, \dots, -1)$. This determines a basis $\{\mathrm{id}, e_{a_1} \dots e_{a_n}\}$ where $a_1 < \cdots < a_n$ and $n = 1, \dots, p+q$ of $\mathrm{Cl}(V, \eta)$. Pick any basis vector e_a, then this guarantees that there is a unique expansion

$$x_0 = a + e_a b \tag{1.38}$$

where a is even, b is odd and both do not contain e_a. Given the expansion of x_0, the equation $x_0 v = v x_0$ then decomposes according to parity into two more equations. These are

$$av = va \,, \tag{1.39}$$
$$e_a b v = v e_a b \,. \tag{1.40}$$

Now set $v = e_a$. Since b does not contain e_a and orthogonality means $e_a e_b = -e_b e_a$ when $a \neq b$, then $b e_a = -e_a b$ since b is odd. The second equation then becomes

$$-(e_a)^2 b = (e_a)^2 b \,, \tag{1.41}$$

where $(e_a)^2 = \pm 1 \neq 0$, which sets $b = 0$. Since the choice of e_a in the expansion of x_0 was arbitrary, this means that $x_0 = \lambda\,\mathrm{id}$ for some $\lambda \in \mathbb{R}$. The same tricks applied to $-x_1 v = v x_1$ also show that $x_1 = 0$. Hence, $x = x_0 = \lambda\,\mathrm{id}$ and since x is a unit, $\lambda \in \mathbb{R}^\times$ is non-zero. □

Now we can show that the Clifford norm is well-behaved on the Lipschitz–Clifford group. It is useful to note that α, τ being (anti)automorphisms guarantees that $\alpha(x^{-1}) = \alpha(x)^{-1}$ and the same for τ.

Proposition 1.7. *The following are true*

1. *If $x \in \Gamma(V, \eta)$, then $N(x) \in \ker(\widetilde{\mathrm{Ad}}) = \mathbb{R}^\times \cdot \mathrm{id}$*
2. *The restriction $N : \Gamma(V, \eta) \to \mathbb{R}^\times$ is a group homomorphism.*
3. *If $x \in \Gamma(V, \eta)$, then $N(\alpha(x)) = N(x)$.*

Proof. We check 1. If $x \in \Gamma(V, \eta)$, then $\tau(\widetilde{\mathrm{Ad}}(x)v) = \widetilde{\mathrm{Ad}}(x)v$ for all $v \in V$. On the otherhand

$$\tau(\widetilde{\mathrm{Ad}}(x)v) = \tau(\alpha(x)vx^{-1}) = \tau(x^{-1})v\kappa(x) = \widetilde{\mathrm{Ad}}(\kappa(x)^{-1})v \tag{1.42}$$

Thus, $\widetilde{\mathrm{Ad}}(x^{-1}\kappa(x^{-1}))v = \widetilde{\mathrm{Ad}}(N(x^{-1}))v = v$. So $N(x^{-1})$ is indeed in the kernel $\ker(\widetilde{\mathrm{Ad}}) = \mathbb{R}^\times {\cdot} \mathrm{id}$.

We check 2. Indeed,

$$N(xy) = xy\kappa(xy) = xN(y)\kappa(x) = N(x)N(y) \tag{1.43}$$

because $N(x) \in \mathbb{R}^\times \cdot \mathrm{id}$.

We check 3. Indeed,

$$N(\alpha(x)) = \alpha(x)\alpha(\kappa(x)) = \alpha(x\kappa(x)) = \alpha(N(x)) = N(x) \tag{1.44}$$

because $N(x) \in \mathbb{R}^\times \cdot \mathrm{id}$. □

Now, using these well-behaved properties of N, we can prove that the $\widetilde{\mathrm{Ad}}$-representation of $\Gamma(V, \eta)$ on V not only contains $O(V, \eta)$ in its image, but that each $\widetilde{\mathrm{Ad}}(x)$ is actually orthogonal.

Proposition 1.8. *If $x \in \Gamma(V,\eta)$, then $\widetilde{\mathrm{Ad}}(x) \in O(V,\eta)$ is orthogonal.*

Proof. We will prove that the norm $v \mapsto \eta(v) := \eta(v,v)$ is preserved and then use the polarisation identities to ensure the inner product is preserved.

Let $x \in \Gamma(V,\eta)$ and $v \in V$ such that $\eta(v) \neq 0$, then $v \in \Gamma(V,\eta)$. Moreover,

$$N(\widetilde{\mathrm{Ad}}(x)v) = N(\alpha(x)vx^{-1}) = N(x)N(v)N(x)^{-1} = N(v)\,. \tag{1.45}$$

Where, for any vector $w \in V$ $N(w) = -w^2 = \eta(w)$. Hence, the above shows $\eta(\widetilde{\mathrm{Ad}}(x)v) = \eta(v)$.

We will invoke topology to deal with the remaining non-zero null-vectors $v \in V$ with $\eta(v) = 0$. Such vectors have a unique expansion as $v = v_1 + v_2$ with $\eta(v_1) = -\eta(v_2) \neq 0$ where we use the usual decomposition $V = V_1 \oplus V_2$. Hence they can also be expressed as $v = \lim_{\varepsilon\to 0} v_\varepsilon$ where $v_\varepsilon = v_1 + (1+\varepsilon)v_2$ and $\eta(v_\varepsilon) = \varepsilon^2\eta(v_2) \neq 0$. Thus, we can write

$$\lim_{\varepsilon\to 0} N(\widetilde{\mathrm{Ad}}(x)v_\varepsilon) = \lim_{\varepsilon\to 0} N(v_\varepsilon)\,. \tag{1.46}$$

Now consider N restricted to V, as mentioned above, it is proportional to the quadratic form η, hence it is continuous. Moreover, linear maps (like $\widetilde{\mathrm{Ad}}(x)$) are continuous on V. Thus,

$$N(\widetilde{\mathrm{Ad}}(x)v) = N(v) \tag{1.47}$$

holds for null-vectors $v \neq 0$ with $\eta(v) = 0$. So the norms of all vectors in V are preserved. □

By putting the various bits of information together we have the short exact sequence

$$\boxed{1 \longrightarrow \mathbb{R}^\times \longrightarrow \Gamma(V,\eta) \xrightarrow{\widetilde{\mathrm{Ad}}} O(V,\eta) \longrightarrow 1\,.} \tag{1.48}$$

From here we can make an argument to explicitly characterise the Clifford–Lipschitz group $\Gamma(V,\eta)$.

Proposition 1.9. *The Clifford–Lipschitz group $\Gamma(V,\eta)$ can be equivalently defined as*

$$\Gamma(V,\eta) := \{\lambda v_1 \dots v_r \mid \eta(v_i) \neq 0\,, \lambda \in \mathbb{R}^\times\}\,. \tag{1.49}$$

In geometric algebra, this new redefinition is called the 'versor group'.

Proof. For now, let $\Gamma(V,\eta)$ take its usual definition and let $G := \{\lambda v_1 \dots v_r \mid \eta(v_i) \neq 0\,, \lambda \in \mathbb{R}^\times\}$. It is clear that $G \subset \Gamma(V,\eta)$.

Take $x \in \Gamma(V,\eta)$, then, because $\widetilde{\mathrm{Ad}}(x)$ is orthogonal, Cartan–Dieudonneé guarantees that there exists a product of reflections $y \in G$ such that $\widetilde{\mathrm{Ad}}(x) = \widetilde{\mathrm{Ad}}(y)$. Hence $xy^{-1} \in \ker \widetilde{\mathrm{Ad}} = \mathbb{R}^\times$. Thus, $x = \lambda y$ for some $\lambda \in \mathbb{R}^\times$. □

Finally we end with another characterisation of the Clifford–Lipschitz group $\Gamma(V,\eta)$.

Lemma 1.10. *The following are true*

1. *For any $x \in \mathrm{Cl}(V,\eta)$, if $\tau(x) = x$, then $x \in \mathbb{R} \oplus V$.*

2. *Let x be a unit. If $N(x) \in \mathbb{R}^\times \cdot \mathrm{id}$ is a non-zero scalar, then $\widetilde{\mathrm{Ad}}(x)v \in V$.*

Proof. We check 1. Indeed, fix a basis e_a of V which diagonalises $\eta_{ab} = \mathrm{diag}(1,\dots,-1)$. As τ is linear it suffices to look at its action on the basis $\{\mathrm{id}, e_{a_1} \dots e_{a_n}\}$ where $a_1 < \dots < a_n$ and $n = 1,\dots,p+q$ of $\mathrm{Cl}(V,\eta)$. Clearly, only the basis vectors id, e_a are preserved. Hence, $\tau(x) = x$ implies $x \in \mathbb{R} \oplus V$.

We check 2. We have $x^{-1} = \lambda^{-1}\kappa(x)$ for some real $\lambda \neq 0$. Thus, any $v \in V$ obeys

$$\widetilde{\mathrm{Ad}}(x)v = \alpha(x)vx^{-1} = -\lambda^{-1}\alpha(xv\tau(x))\,. \tag{1.50}$$

Moreover

$$\tau(\widetilde{\mathrm{Ad}}(x)v) = -\lambda^{-1}\alpha(\tau(xv\tau(x))) = -\lambda^{-1}\alpha(xv\tau(x)) = \widetilde{\mathrm{Ad}}(x)v \tag{1.51}$$

Finally, $\widetilde{\mathrm{Ad}}(x)v$ must be in V by parity. □

As a result of the proposition and lemma we now have a nice theorem.

Theorem 1.11. *The following definitions are all equivalent*

1. $\Gamma(V,\eta) := \{x \in \mathrm{Cl}(V,\eta)^\times \mid \widetilde{\mathrm{Ad}}(x)(v) \in V \quad \text{for all } v \in V\}\,.$
2. $\Gamma(V,\eta) := \{\lambda v_1 \ldots v_r \mid \eta(v_i) \neq 0\,, \lambda \in \mathbb{R}^\times\}\,.$
3. $\Gamma(V,\eta) := \{x \in \mathrm{Cl}(V,\eta)^\times \mid N(x) \in \mathbb{R}^\times \cdot \mathrm{id}\}\,.$

1.3.4 Pin and spin groups

We are now ready to define pin and spin groups. We will make an effort to relate to other definitions of them.

Definition 1.12. The *pin group* is defined as any of the equivalent definitions (by Theorem 1.11)

$$\mathrm{Pin}(V,\eta) := \{x \in \Gamma(V,\eta) \mid N(x) = \pm 1\}\,, \tag{1.52}$$
$$\mathrm{Pin}(V,\eta) := \{x \in \mathrm{Cl}(V,\eta)^\times \mid N(x) = \pm 1\}\,, \tag{1.53}$$
$$\mathrm{Pin}(V,\eta) := \{\pm v_1 \ldots v_r \mid \eta(v_i) = \pm 1\}\,. \tag{1.54}$$

The *spin group* is defined as $\mathrm{Spin}(V,\eta) := \mathrm{Pin}^0(V,\eta)$, i.e. the even part. The *orthochronous spin group* is defined as $\mathrm{Spin}^+(V,\eta) := \{x \in \mathrm{Spin}(V,\eta) \mid N(x) = 1\}$. It also goes by the name 'rotor group' in geometric algebra.

General comments Indeed, the last definition makes sense because any product of vectors can be written as a product of unit vectors multiplied by some λ, the possible values of λ which satisfy $N(x) = \pm 1$ is the the set of real square roots of ± 1, which is exactly ± 1. When we define the complex spin group this will cause annoyances because ± 1 has square roots $\pm 1, \pm i$.

Note that $N(x) = \pm 1$ is indeed needed to define pin and spin groups in general. The plus or minus one is indicative of the presence of vectors v with negative norm $N(v) = -\eta(v) < 0$. This of course occurs for a non-degenerate indefinite quadratic form η. It also occurs for *negative-definite* quadratic forms. Indeed, in our $V = V_1 \oplus V_2$ convention with $\eta|_{V_1}$ positive-definite and $\eta|_{V_2}$ negative-definite, along with our Clifford algebra convention

$$vw + wv = -2\eta(v,w) \tag{1.55}$$

we find a unit time-like vector has $N(v_1) = -v_1^2 = \eta(v_1) = 1$ whereas a unit space-like vector has $N(v_2) = -v_2^2 = \eta(v_2) = -1$. Setting $V_1 = 0$ so that η is negative-definite then reveals that $N(v_2) = -1$ still occurs.

The only time when $N(x) = 1$ is sufficient to describe $\mathrm{Pin}(V,\eta)$ (in our conventions) is when $V = V_1$.

Setting $N(x) = 1$ is still useful though (as is evident in $\mathrm{Spin}^+(V,\eta)$'s definition). Indeed, in our conventions, setting $N(x) = 1$ forces $\widetilde{\mathrm{Ad}}(x)$ to always involve an *even number of time-like reflections*, hence temporal orientation is preserved and x is called *orthochronous*. As a consequence, if $\widetilde{\mathrm{Ad}}(x) \in O(V_1, \eta|_{V_1})$ only affects V_1, then $N(x) = 1$ iff $\det(\widetilde{\mathrm{Ad}}(x)) = 1$.

In particular, this means we have very explicit realisations of the groups as

$$\mathrm{Pin}(V,\eta) := \{\pm v_1 \ldots v_r \mid \eta(v_i) = \pm 1\}\,, \tag{1.56a}$$
$$\mathrm{Spin}(V,\eta) := \{\pm v_1 \ldots v_{2r} \mid \eta(v_i) = \pm 1\}\,, \tag{1.56b}$$
$$\mathrm{Spin}^+(V,\eta) := \{\pm v_1 \ldots v_{2r} \mid \eta(v_i) = \pm 1,\ \text{the number of } v_i \text{ such that } \eta(v_i) = 1 \text{ is even}\}\,. \tag{1.56c}$$

Definition 1.13. The *orthochronous special orthogonal group* is defined as $SO^+(V,\eta) := \widetilde{\mathrm{Ad}}(\mathrm{Spin}^+(V,\eta))$.

From the above explicit realisation of $\mathrm{Spin}^+(V,\eta)$, it is obvious that $SO^+(V,\eta)$ is a subgroup of $SO(V,\eta)$ generated by products of reflections with an even number of time-like reflections and an even number of space-like reflections.

From here onwards it is easier to state/read the results by using the signature (p,q) of (V,η), so we state the following in that notation.

Theorem 1.14. *The following results hold*

1. *The following are exact sequences*

$$1 \longrightarrow \{-1,1\} \longrightarrow \mathrm{Pin}(p,q) \xrightarrow{\widetilde{\mathrm{Ad}}} O(p,q) \longrightarrow 1\,, \tag{1.57}$$

$$1 \longrightarrow \{-1,1\} \longrightarrow \mathrm{Spin}(p,q) \xrightarrow{\widetilde{\mathrm{Ad}}} SO(p,q) \longrightarrow 1\,, \tag{1.58}$$

$$1 \longrightarrow \{-1,1\} \longrightarrow \mathrm{Spin}^+(p,q) \xrightarrow{\widetilde{\mathrm{Ad}}} SO^+(p,q) \longrightarrow 1\,, \tag{1.59}$$

and $\widetilde{\mathrm{Ad}}$ *is a (double) covering map in each case.*

2. $\mathrm{Spin}^+(p,q) = \mathrm{Spin}(p,q)$ *iff* η *is positive-definite or negative definite. The same holds for* $SO^+(p,q) = SO(p,q)$.

3. $\mathrm{Spin}(p,q) \cong \mathrm{Spin}(q,p)$, *but not in general for pin groups.*

4. *The orthochronous special orthogonal group* $SO^+(p,q)$ *is connected. Hence, it is the identity connected component of the orthogonal groups.*

5. *The orthochronous spin group* $\mathrm{Spin}^+(p,q)$ *is connected except in the cases* $(p,q) = (1,1), (1,0), (0,1)$ *where instead* $\mathrm{Spin}^+(1,1) \cong \mathbb{R}^\times$, *and* $\mathrm{Spin}^+(1,0) = \mathrm{Spin}^+(0,1) = \{-1,1\}$. *Hence, it is (mostly) the identity connected component of the pin and spin groups and, in the cases where it is, all groups provide a non-trivial double cover of their respective orthogonal groups.*

6. *For definite signatures,* $\pi_1(SO(1)) = 1$, $\pi_1(SO(2)) \cong \mathbb{Z}$, $\pi_1(SO(n)) \cong \mathbb{Z}_2$ *for* $n \geq 3$. *Moreover, for indefinite signatures* (p,q) *with* $p \geq 1$ *and* $q \geq 1$

$$\pi_1(SO^+(p,q)) \cong \begin{cases} 1 & \textit{if } (p,q) = (1,1)\,, \\ \mathbb{Z} & \textit{if } (p,q) = (1,2)\,, \\ \mathbb{Z}_2 & \textit{if } p = 1 \textit{ and } q \geq 3\,, \\ \mathbb{Z}\times\mathbb{Z} & \textit{if } (p,q) = (2,2)\,, \\ \mathbb{Z}\times\mathbb{Z}_2 & \textit{if } p = 2 \textit{ and } q \geq 3\,, \\ \mathbb{Z}_2\times\mathbb{Z}_2 & \textit{if } p,q \geq 3\,, \end{cases} \tag{1.60}$$

keeping in mind that $O(p,q) = O(q,p)$. *Consequently, due to the non-trivial double covering* $\pi_1(\mathrm{Spin}(2)) \cong \mathbb{Z}$, $\pi_1(\mathrm{Spin}(n)) \cong 1$ *for* $n \geq 3$. *For indefinite signatures*

$$\pi_1(\mathrm{Spin}^+(p,q)) \cong \begin{cases} 1 & \textit{if } (p,q) = (1,1)\,, \\ \mathbb{Z} & \textit{if } (p,q) = (1,2)\,, \\ 1 & \textit{if } p = 1 \textit{ and } q \geq 3\,, \\ \mathbb{Z}\times\mathbb{Z} & \textit{if } (p,q) = (2,2)\,, \\ \mathbb{Z}(\textit{or maybe } \mathbb{Z}\times\mathbb{Z}_2) & \textit{if } p = 2 \textit{ and } q \geq 3\,, \\ \mathbb{Z}_2 & \textit{if } p,q \geq 3\,, \end{cases} \tag{1.61}$$

keeping in mind that $\mathrm{Spin}(p,q) = \mathrm{Spin}(q,p)$. *Hence, the covering is typically not universal.*

Proof. We check 1.

Since $\ker \widetilde{\mathrm{Ad}} = \mathbb{R}^\times \cdot \mathrm{id}$ for the Clifford–Lipschitz group, then it is clear that the kernel must restrict to the subgroup of $\mathbb{R}^\times \cdot \mathrm{id}$ contained in the pin group. It is not hard to see that if

$x = \lambda\,\mathrm{id}$, then $N(x) = \lambda^2 = \pm 1$ has solutions $\lambda = \pm 1$. Hence, when restricted to the pin group, $\ker \widetilde{\mathrm{Ad}} = \{\pm 1\}$. The same holds for the spin and orthochronous spin subgroups. Hence, each point $p \in \mathrm{im}\,\widetilde{\mathrm{Ad}}$ has a preimage of 2 points $\{x, -x\}$ in the pin, spin and orthochronous spin groups. We skip checking this is a local homeomorphism, but it indeed is.

We check 2.

This is immediate from (1.56). Indeed, $\mathrm{Spin}^+(V, \eta)$ contains even products of unit vectors which also have an even number of unit time-like vectors. If there are only time-like vectors (positive-definite), then $\mathrm{Spin}^+(p, 0) = \mathrm{Spin}(p, 0)$ because $\mathrm{Spin}(p, 0)$ just contains even products of unit vectors. The same holds for the negative-definite case. If η is indefinite, i.e. p, q are both non-zero, then a product of a unit time-like and space-like vector is in $\mathrm{Spin}(p, q)$, but not $\mathrm{Spin}^+(p, q)$, so they are not equal. The special orthogonal version is the same argument, just with reflections (take the image under $\widetilde{\mathrm{Ad}}$).

We check 3.

This is easier to check later when we classify real Clifford algebras. In particular, we will find a non-canonical algebra isomorphism $\mathrm{Cl}^0(p, q) \cong \mathrm{Cl}^0(q, p)$ which descends to a group isomorphism $\mathrm{Spin}(p, q) \cong \mathrm{Spin}(q, p)$.

We check 4.

Fix a decomposition $V = V_1 \oplus V_2$, and note that $\sigma_v \sigma_w = \sigma_{\sigma_v(w)} \sigma_v$ where $\eta(w) = \eta(\sigma_v(w))$ preserves that w is space-like ($\eta(w) < 0$) or time-like ($\eta(w) > 0$). This ensures any product of reflections $\sigma_{v_1} \ldots \sigma_{v_r}$ can be rewritten with time-like reflections grouped on the left and space-like reflections grouped on the right. Now, we can of course make a path between any two vectors v, w by

$$\gamma(t) = (1 - t)v + tw\,. \tag{1.62}$$

However, if $\eta(v) > 0$ and $\eta(w) < 0$ intermediate value theorem dictates that $\eta(\gamma(t)) = 0$ for some t and hence the curve would *not* always have a well-defined reflection σ. Now, any time-like vector has a unique decomposition $v = v_1 + v_2$, $\eta(v) > 0$ and can be continuously joined to v_1 by

$$\gamma(t) = v_1 + (1 - t)v_2 \tag{1.63}$$

where $\eta(\gamma(t)) = \eta(v_1) + (1 - t)^2 \eta(v_2) \geq \eta(v) > 0$. Moreover, $\eta|_{V_1}$ is positive-definite, so any vectors within it may be connected by a path and retain positive norm. Hence, by continuity, there is a path $\gamma(t)$ in $O(p, q)$ between any two time-like reflections. The same arguments hold for space-like reflections. Using that $\sigma_v^2 = 1$ is idempotent for any $v \in V$ such that $v \neq 0$, it is clear that any element of $SO^+(p, q)$ is connected to the identity precisely because such elements have an even number of time-like reflections and space-like reflections.

We check 5.

In the two 1-dimensional cases, the definitions themselves (1.56) ensures $\mathrm{Spin}^+(1, 0) = \mathrm{Spin}^+(0, 1) = \{-1, 1\}$. In the $\mathrm{Spin}^+(1, 1)$ case, fix an orthonormal basis $\{e_1, e_2\}$ of V with $\eta(e_1) = -\eta(e_2) = 1$. We claim that every element $x \in \mathrm{Spin}^+(1, 1)$ is of the form

$$x = \pm e^{te_1e_2} = \pm(\cosh(t) + \sinh(t)e_1e_2) \tag{1.64}$$

for some $t \in \mathbb{R}$. This is not hard to check as any element of $\mathrm{Cl}^0(1, 1)$ has a unique expansion $x = a + be_1e_2$ and $N(x) = a^2 - b^2$. Hence, $x \in \mathrm{Spin}^+(1, 1)$ iff $a^2 - b^2 = 1$. This has a general solution $x = \pm\sqrt{1 + b^2} + be_1e_2$ for $b \in \mathbb{R}$. The invertible relation $b = \sinh(t)$ gives the result. A group isomorphism $\mathrm{Spin}^+(1, 1) \to \mathbb{R}^\times$ is then given by $\pm e^{te_1e_2} \mapsto \pm e^t$.

Suppose $\dim V = p + q \geq 2$ and $(p, q) \neq (1, 1)$. We of course know that $SO^+(p, q)$ is connected and $\widetilde{\mathrm{Ad}} : \mathrm{Spin}^+(p, q) \to SO^+(p, q)$ is a double covering with $\ker \widetilde{\mathrm{Ad}} = \{-1, 1\}$. Hence $\mathrm{Spin}^+(p, q)$ is connected iff ± 1 live in the same connected component. Due to our (p, q) assumptions we can find orthonormal vectors e_1, e_2 on which η is positive-definite or negative definite. That is $(\eta(e_1), \eta(e_2)) = (1, 1), (-1, -1)$. W.l.o.g. take the $(1, 1)$ case and define a path γ in $\mathrm{Spin}^+(p, q)$ by

$$\gamma(t) = (\cos te_1 + \sin te_2)(\cos te_1 - \sin te_2) \tag{1.65}$$

Clearly, $\gamma(0) = e_1^2 = -\eta(e_1) = -1$ and $\gamma(\frac{\pi}{2}) = -e_2^2 = \eta(e_2) = 1$ gives a path from -1 to 1. This also works for the $(-1,-1)$ case, so ± 1 are indeed connected. Note it would *fail* if we tried it for the $(1,-1)$ case as $\eta((\cos(\frac{\pi}{4})e_1 + \sin(\frac{\pi}{4})e_2)) = 0$ has zero norm.

We check 6.

We give a sketch. First $SO(1) = 1$ is just the trivial group with a single point given by the identity element, so $\pi_1(SO(1)) = 1$ is trivial. Second $SO(2) \cong U(1) \cong S^1$ is a 'nice' topological space[4] and it is universally covered by $\mathbb{R}$ with fiber $\mathbb{Z}$, so $\pi_1(SO(2)) \cong \mathbb{Z}$. In the case of $SO(3)$, one can check explicitly using that $\mathrm{Spin}(3) \cong SU(2) \cong S^3$. Then use that $\pi_1(S^3) = 1$ is trivial so that Spin(3) is a universal double cover, which means $\pi_1(SO(3)) \cong \mathbb{Z}_2$. To deal with $n \geq 3$ in general we use the fibration $SO(n) \to SO(n+1) \to S^n$[5] and its long exact sequence in homotopy groups. Indeed, the sequence

$$\pi_2(S^n) \longrightarrow \pi_1(SO(n)) \longrightarrow \pi_1(SO(n+1)) \longrightarrow \pi_1(S^n) \tag{1.66}$$

is exact and $\pi_1(S^n) = \pi_2(S^n) = 1$ are trivial since $n \geq 3$. Thus, $\pi_1(SO(n)) \cong \pi_1(SO(n+1))$ for $n \geq 3$.

In the indefinite case we use the fact that $SO^+(p,q)$ has a maximal compact subgroup given by $SO(p) \times SO(q)$ and that Cartan's decomposition theorem for Lie groups (Theorem 1.15) ensures a Lie group deformation retracts onto a maximal compact subgroup. Thus, we get the rest from

$$\pi_1(SO^+(p,q)) \cong \pi_1(SO(p)) \times \pi_1(SO(q)). \tag{1.67}$$

Assume $(p,q) \neq (1,1), (1,0), (0,1)$ as we can deal with these cases manually. We can use the long exact sequence in homotopy to get an exact sequence

$$\pi_1(\mathbb{Z}_2) \longrightarrow \pi_1(\mathrm{Spin}^+(p,q)) \xrightarrow{A} \pi_1(SO^+(p,q)) \xrightarrow{B} \pi_0(\mathbb{Z}_2) \xrightarrow{C} \pi_0(\mathrm{Spin}^+(p,q)), \tag{1.68}$$

$$1 \longrightarrow \pi_1(\mathrm{Spin}^+(p,q)) \xrightarrow{A} \pi_1(SO^+(p,q)) \xrightarrow{B} \mathbb{Z}_2 \xrightarrow{C} 1, \tag{1.69}$$

where $\pi_1(\mathbb{Z}_2) = \pi(\mathrm{pt}) = 1$ and $\pi_0(\mathbb{Z}_2) = \mathbb{Z}_2$ and $\pi_0(\mathrm{Spin}^+(p,q)) = 1$. Hence, $\ker A = 1$ is trivial so that A is injective, and $\pi_1(\mathrm{Spin}^+(p,q)) \cong \operatorname{im} A = \ker B$. We also have that B is surjective since $\operatorname{im} B = \ker C = \mathbb{Z}_2$.

In the case $\pi_1(SO^+(p,q) \cong \mathbb{Z}_2$, we find that $\pi_1(\mathrm{Spin}^+(p,q)) \cong \operatorname{im} A = \ker B = 1$, so $\mathrm{Spin}^+(p,q)$ is simply connected in this case.

In the case $\pi_1(SO^+(p,q)) \cong \mathbb{Z}$, we find that $\pi_1(\mathrm{Spin}^+(p,q)) \cong \operatorname{im} A = \ker B = 2\mathbb{Z} \cong \mathbb{Z}$. Hence both groups have the same fundamental group.

In the case $\pi_1(SO^+(p,q)) \cong \mathbb{Z} \times \mathbb{Z}$, we find that $\pi_1(\mathrm{Spin}^+(p,q)) \cong \operatorname{im} A = \ker B \cong \mathbb{Z} \times \mathbb{Z}$. Hence both groups have the same fundamental group.

In the case $\pi_1(SO^+(p,q)) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, we find that $\pi_1(\mathrm{Spin}^+(p,q)) \cong \operatorname{im} A = \ker B \cong \mathbb{Z}_2$. Hence we have dropped a factor of $\mathbb{Z}_2$.

In the remaining case $\pi_1(SO^+(p,q)) \cong \mathbb{Z} \times \mathbb{Z}_2$ we don't have enough info because the kernel of a surjective map $\mathbb{Z} \times \mathbb{Z}_2 \to \mathbb{Z}_2$ is non-unique, in fact there are 3 cases $\mathbb{Z} \times \mathbb{Z}_2, \mathbb{Z}, \mathbb{Z}_2$. In principle one could find out by determining A explicitly from $\widetilde{\mathrm{Ad}}$. Indeed, actually

$$\mathrm{Spin}^+(p,q) \xrightarrow{\widetilde{\mathrm{Ad}}} SO^+(p,q) \tag{1.70}$$

is a non-trivial double cover over the subgroups $SO(p)$ and $SO(q)$. More precisely, the preimages of these subgroups under $\widetilde{\mathrm{Ad}}$ must be $\mathrm{Spin}(p)$ and $\mathrm{Spin}(q)$. Hence,

$$\pi_1(\mathrm{Spin}^+(p,q)) \xrightarrow{A} \pi_1(SO^+(p,q)) \xrightarrow{\sim} \pi_1(SO(p)) \times \pi_1(SO(q)) \tag{1.71}$$

[4]This has to do with path-connected, locally path-connected, semi-locally path connected etc. Such requirements are fulfilled by connected manifolds (like the circle S^1).

[5]This is in fact a principal bundle with structure group $SO(n)$. One can realise this fibration easily with $SO(n+1) \to S^n \subseteq \mathbb{R}^{n+1}$ by $B \mapsto Be_{n+1}$ so that a preimage of any point is diffeomorphic to $SO(n)$.

involves non-trivial maps

$$\pi_1(\mathrm{Spin}^+(p,q)) \longrightarrow \pi_1(SO(p)) \cong \mathbb{Z}\,, \qquad \pi_1(\mathrm{Spin}^+(p,q)) \longrightarrow \pi_1(SO(q)) \cong \mathbb{Z}_2 \tag{1.72}$$

Thus, the $\mathbb{Z}_2$ case is not possible. Considering the individual preimages under $\widetilde{\mathrm{Ad}}$ explicitly, we find that $\pi_1(\mathrm{Spin}^+(p,q)) \cong \mathbb{Z}$. □

In the above proof have made use of the following theorem so that Lie groups deformation retract onto maximal compact subgroups.

Theorem 1.15 (Cartan). *A connected real Lie group G is diffeomorphic to $K \times \mathbb{R}^n$ where K is a maximal compact subgroup of G. Moreover, all maximal compact subgroups are conjugate to K.*

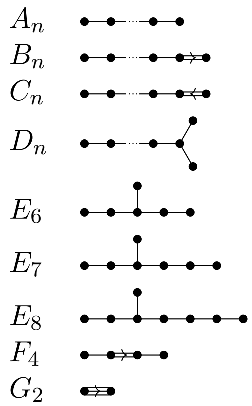
1.3.5 A brief look at representation theory—from complex simple Lie algebra p.o.v.

We look at only *finite dimensional complex* representations here. We give a quick rundown of some rep theory results on semisimple Lie algebras.

The classical complex Lie algebras come in the four families

$$A_n := \mathfrak{sl}(n+1, \mathbb{C})\,, \tag{1.73}$$
$$B_n := \mathfrak{so}(2n+1, \mathbb{C})\,, \tag{1.74}$$
$$C_n := \mathfrak{sp}(2n, \mathbb{C})\,, \tag{1.75}$$
$$D_n := \mathfrak{so}(2n, \mathbb{C})\,, \tag{1.76}$$

and, along with the exceptional algebras E_6, E_7, E_8, F_4, G_2, they list all the complex simple Lie algebras (up to low-dimensional cases $D_1 = \mathfrak{so}(2,\mathbb{C}), D_2 = \mathfrak{so}(4,\mathbb{C})$, which are not simple). They have Dynkin diagrams



which correspond to a graphical realisation of the simple roots and their relations via the Cartan matrix $A = (a_{ij})$—a node for each simple root and an edge between nodes based on entries in the Cartan matrix. A key point is that the *fundamental weights* ω_i are related to the simple roots by α_i by $\alpha_i = a_{ij}\omega_j$. Every weight λ corresponds to irrep with highest weight λ and *all* irreps arise this way. The fundamental weights have the property that *any* weight λ is a unique non-negative integer linear combination of them.

Noting that, given two highest weight irreps $V_{\lambda_1}, V_{\lambda_2}$, the tensor product $V_{\lambda_1} \otimes V_{\lambda_2}$ contains the irrep $V_{\lambda_1+\lambda_2}$, we can see that all irreps can be realised as subreps of a tensor product of the fundamental reps V_{ω_i}. Finally, Weyl's theorem on complete reducibility ensures that all reps can be decomposed into a sum of irreps, so that we can construct *all* reps using only V_{ω_i}.

Theorem 1.16 (Weyl's theorem on complete reducibility). *Consider a semisimple Lie algebra over a field k of characteristic zero. All its finite dimensional reps over k are completely reducible.*

Proof. Use Whitehead's first lemma and follow Wiki. Alternatively, over $k = \mathbb{C}$, one can leverage compact real forms in the spirit of Weyl. □

This theory holds generally for complex semisimple Lie algebras (which are direct sums of the simple ones). However, we are interested in the case of the special orthogonal algebras $\mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C}), \mathfrak{so}(2n, \mathbb{C}) = B_n, D_n$. Due to the double covering and connectedness results, we find that all the following real Lie algebras are the same

$$\mathfrak{pin}(p,q) \cong \mathfrak{spin}(p,q) \cong \mathfrak{spin}^+(p,q) \cong \mathfrak{so}(p,q) \tag{1.77}$$

Moreover, we can complexify to get $\mathfrak{so}(p,q) \otimes \mathbb{C} \cong \mathfrak{so}(p+q, \mathbb{C})$. The simple result below then determines the structure of complex reps of $\mathfrak{so}(p,q)$.

Proposition 1.17. *Let $\mathfrak{g}$ be a real Lie algebra, then any complex rep V gives a complex rep of its complexification $\mathfrak{g}_{\mathbb{C}}$. Moreover, by restriction we can take a complex rep of $\mathfrak{g}_{\mathbb{C}}$ to one for $\mathfrak{g}$. These operations are inverse.*

Proof. Given a complex rep $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ we need only define

$$\rho_{\mathbb{C}}(x + iy) := \rho(x) + i\rho(y) \tag{1.78}$$

for each $x, y \in \mathfrak{g}$ to lift it to a rep of $\mathfrak{g}_{\mathbb{C}}$ because ρ is $\mathbb{R}$-linear and already preserves $\mathfrak{g}$'s Lie bracket and

$$[x + iy, z + iw]_{\mathbb{C}} := [x, z] - [y, w] + i([x, w] + [y, z])\,. \tag{1.79}$$

Given a complex rep of $\mathfrak{g}_{\mathbb{C}}$ we simply restrict to $\mathfrak{g} \subseteq \mathfrak{g}_{\mathbb{C}}$ and $\mathbb{C}$-linearity and $[-,-]_{\mathbb{C}}$ preservation immediately restrict to $\mathbb{R}$-linearity and $[-,-]$ preservation.

These operations obviously undo eachother, hence are inverse. □

The key point of all of this is that the fundamental rep(s) corresponding to the right-most nodes on the Dynkin diagrams B_n, D_n are the *fundamental spinor rep(s)*, i.e. the complex spin-$\frac{1}{2}$ representation(s) of $\mathfrak{so}(p+q, \mathbb{C})$ and $\mathfrak{so}(p,q)$. In physics these would be called the *complex Weyl spinor rep(s)*.

Moreover, these spinor rep(s) can be realised as reps of the spin group $\mathrm{Spin}^+(p,q)$, even in the cases where it is *not* simply connected. This could be seen as being due to the complexification $\mathrm{Spin}^+(p+q, \mathbb{C})$ being simply connected, and all of complex rep theory depending only on complexified versions of groups/algebras.[6] In case this is unclear, we are referencing Lie's three theorems relating Lie algebras and Lie groups, in particular that a Lie algebra is (in some precise way) the same information as a connected, simply connected Lie group.

The non-spinor fundamental representations all arise in the following way (from Fulton-Harris section 19.2).

Theorem 1.18. *The following hold for $n \geq 0$ (maybe issues with $D_1 = \mathfrak{so}(2, \mathbb{C}), D_2 = \mathfrak{so}(4, \mathbb{C})$):*

1. *Let $V = \mathbb{C}^{2n}$ be the defining/standard representation of $\mathfrak{so}(2n, \mathbb{C})$. All non-spinor fundamental representations are $V, \Lambda^2 V, \ldots, \Lambda^{n-2} V$. There are two remaining fundamental spinor representations and arise as the unique two irreps of the even part $\mathrm{Cl}(2n)^0 \supseteq \mathfrak{so}(2n, \mathbb{C})$ of the complex Clifford algebra.*

2. *Let $V = \mathbb{C}^{2n+1}$ be the defining/standard representation of $\mathfrak{so}(2n+1, \mathbb{C})$. All non-spinor fundamental representations are $V, \Lambda^2 V, \ldots, \Lambda^{n-1} V$. There is one remaining fundamental spinor representation which arises as the unique irrep of the even part $\mathrm{Cl}(2n+1)^0 \supseteq \mathfrak{so}(2n+1, \mathbb{C})$ of the complex Clifford algebra.*

[6] One defines the complex spin groups much the same way as we define the real ones. Just follow the earlier processes, but for $k = \mathbb{C}$.

The proof is more of an exercise in Lie algebra theory, so we don't give it here. The main thing is that these unique irreps of the even part of complex Clifford algebra give the fundamental spinor irreps through a natural embedding $\mathfrak{so}(n,\mathbb{C}) \subseteq \mathbb{C}\text{l}(n)^0$.

We briefly describe this embedding here and, more generally, the embedding $\mathfrak{so}(p,q) \subseteq \text{Cl}(p,q)^0$. Recall that the special orthogonal algebra can be defined by differentiating the condition for the orthogonal group. That is $\eta(\Lambda v, \Lambda w) = \eta(v,w)$ becomes

$$\frac{d}{dt}\eta(e^{Xt}v, e^{Xt}w)|_{t=0} = \eta(Xv,w) + \eta(v,Xw) = 0\,, \tag{1.80}$$

and $\mathfrak{so}(p,q)$ is the space of all such linear maps X. Indeed, one can check that λX and $[X,Y] = XY - YX$ obey the condition if X, Y do.

Now consider the space of 2-forms $\Lambda^2(V)$ endowed with the following Lie bracket,

$$[a\wedge b, c\wedge d] = 2\eta(b,c)a\wedge d - 2\eta(b,d)a\wedge c - 2\eta(a,d)c\wedge b + 2\eta(a,c)d\wedge b\,. \tag{1.81}$$

We claim that the map $\Lambda^2(V) \to \mathfrak{so}(p,q)$ given by

$$a\wedge b(v) = 2(\eta(b,v)a - \eta(a,v)b)\,, \tag{1.82}$$

is an isomorphism. Indeed, one can check that this is linear, $a\wedge b$ acts linearly on V and the Lie bracket is preserved. Then one just uses some basis to check it is an isomorphism.

Finally, we the linear map $\Lambda^2(V) \to \text{Cl}(p,q)^0$ defined by

$$a\wedge b \mapsto ab - ba\,, \tag{1.83}$$

gives the desired embedding of $\mathfrak{so}(p,q)$ into $\text{Cl}(p,q)^0$. Indeed, this is what one always uses in physics in the basis $e_a \equiv \Gamma_a$ of V where

$$\Gamma_{ab} = \frac{1}{2}(\Gamma_a\Gamma_b - \Gamma_b\Gamma_a)\,. \tag{1.84}$$

1.4 Classification of real Clifford algebras

It is clear that reps of the algebra $\text{Cl}(p,q)$ give reps of its group of units $\text{Cl}(p,q)^\times$ and hence all of its subgroups, like $\text{Pin}(p,q)$, $\text{Spin}(p,q)$, $\text{Spin}^+(p,q)$ by restriction. Similar things can be said about reps of $\text{Cl}^0(p,q)$ giving reps for the even subgroups, like $\text{Spin}(p,q)$ and $\text{Spin}^+(p,q)$.

We will find that the fundamental spinor representation(s), i.e. the complex spin-$\frac{1}{2}$ representation(s) of $\text{Spin}^+(p,q)$ will arise as the complexification of the only irreps of $\text{Cl}^0(p,q)$ (i.e. that even parts of Clifford algebras only have one irrep if $d = p+q$ is odd or two irreps if it is even).

The first step to this result is to classify the Clifford algebras as being matrix algebras or the sum of two matrix algebras. Then we can use ring theory to classify their reps. As a first step we will introduce the so-called *volume element* in analogy with a top-form/volume element of the exterior algebra.

Definition 1.19 (volume element/pseudoscalar)**.** Fix a basis of (V,η) so that $V = \mathbb{R}^{p,q}$ and $d = p+q$, then the *volume element/pseudoscalar* of $\text{Cl}(p,q)$ is $\Gamma_* = \Gamma_{d+1} = e_1 \ldots e_{p+q} = \Gamma_1 \ldots \Gamma_{p+q}$, where we are using the 'gamma matrix' notation[7]. The choice of basis is irrelevant, indeed it can only change Γ_* up to a sign as $\Lambda^n(G) = \det G$.

The volume element lies in the Clifford-Lipschitz group $\Gamma(p,q)$ and $\widetilde{\text{Ad}}(\pm\Gamma_*)$ is always well-defined and corresponds to reflection about all axes (space and time), so it is PT. It also has nice properties.

[7] Note the 'gamma matrices' Γ_a are not necessarily matrices as we haven't specified any rep of $\text{Cl}(p,q)$

Proposition 1.20. *The volume element* Γ_* *satisfies* ($d = p + q$)*:*

$$\Gamma_a \Gamma_* = (-1)^{d-1} \Gamma_* \Gamma_a \,, \tag{1.85}$$

$$(\Gamma_*)^2 = (-1)^{\frac{d(d-1)}{2}+p} \,. \tag{1.86}$$

Equivalently, the odd and even $d = p + q$ *cases are respectively*

$$\Gamma_a \Gamma_* = \Gamma_* \Gamma_a \,, \qquad \Gamma_a \Gamma_* = -\,\Gamma_* \Gamma_a \,, \tag{1.87}$$

$$(\Gamma_*)^2 = (-1)^{\frac{(p-q+1)}{2}} \,, \qquad (\Gamma_*)^2 = (-1)^{\frac{p-q}{2}} \,. \tag{1.88}$$

It is also useful to note that

$$(\Gamma_*)^2 = \begin{cases} 1 & \text{if } 0,3 \equiv p - q \pmod 4 \,, \\ -1 & \text{if } 1,2 \equiv p - q \pmod 4 \,. \end{cases} \tag{1.89}$$

$$(\Gamma_*)^2 = \begin{cases} 1 & \text{if } 0,3,4,7 \equiv p - q \pmod 8 \,, \\ -1 & \text{if } 1,2,5,6 \equiv p - q \pmod 8 \,. \end{cases} \tag{1.90}$$

Proof. Omitted, just amounts to computations using $\Gamma_a \Gamma_b + \Gamma_b \Gamma_a = -2\eta_{ab}\,\mathrm{id}$ with $\eta_{ab} = \mathrm{diag}(1, \ldots, -1)$. □

We observe some low-dimensional examples.

Proposition 1.21. *The following low-dimensional isomorphisms hold:*

1. $\mathrm{Cl}(1,0) \cong \mathbb{C}$,
2. $\mathrm{Cl}(0,1) \cong \mathbb{R} \oplus \mathbb{R}$,
3. $\mathrm{Cl}(2,0) \cong \mathbb{H}$,
4. $\mathrm{Cl}(0,2) \cong \mathrm{Mat}_2(\mathbb{R})$,
5. $\mathrm{Cl}(1,1) \cong \mathrm{Mat}_2(\mathbb{R})$.

Proof. We check 1. This follows from writing $\mathrm{Cl}(1,0)$ w.r.t. the usual basis $\{\mathrm{id}, e_1\}$ with $(e_1)^2 = -\eta(e_1) = -1$.

We check 2. Similar, but now we have basis $\{\mathrm{id}, e_1\}$ with $(e_1)^2 = -\eta(e_1) = 1$. Hence, as a vector space we have $\mathbb{R}^2 = \mathbb{R} \oplus \mathbb{R}$ and then we put some multiplication on it.

We check 3. In this case we have a basis $\{\mathrm{id}, e_1, e_2, e_1 e_2\}$ where $(e_1)^2 = (e_2)^2 = (e_1 e_2)^2 = -1$. Hence, a map to $\mathbb{H}$ is given by $e_1 \mapsto i$, $e_2 \mapsto j$, $e_1 e_2 \mapsto ij = k$.

We check 4. In this case we have a basis $\{\mathrm{id}, e_1, e_2, e_1 e_2\}$ where $(e_1)^2 = (e_2)^2 = 1$ and $(e_1 e_2)^2 = -1$. In this case consider the map

$$\mathrm{id} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \qquad e_1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \qquad e_2 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad e_1 e_2 \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{1.91}$$

Such matrices clearly span $\mathrm{Mat}_2(\mathbb{R})$.

We check 5. In this case we have a basis $\{\mathrm{id}, e_1, e_2, e_1 e_2\}$ with $(e_1)^2 = -1$ and $(e_2)^2 = (e_1 e_2)^2 = 1$. In this case we almost copy $\mathrm{Cl}(0,2)$ and use the map

$$\mathrm{id} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \qquad e_1 \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \qquad e_2 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad e_1 e_2 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{1.92}$$

□

Next we note that larger Clifford algebras can be constructed from smaller ones.

Lemma 1.22. *The following 'periods' hold:*

1. $\mathrm{Cl}(d+2,0) \cong \mathrm{Cl}(0,d) \otimes \mathrm{Cl}(2,0)$,

2. $\mathrm{Cl}(0,d+2) \cong \mathrm{Cl}(d,0) \otimes \mathrm{Cl}(0,2)$,

3. $\mathrm{Cl}(p+1,q+1) \cong \mathrm{Cl}(p,q) \otimes \mathrm{Cl}(1,1)$.

Proof. This is easiest to see at the level of generator mappings. Moreover, I find the 'gamma matrix' notation nicer here.

We check 1. Let $\Gamma'_1, \ldots, \Gamma'_d$ be the generators of $\mathrm{Cl}(0,d)$ and Γ''_1, Γ''_2 be the generators of $\mathrm{Cl}(2,0)$. Define a map $\mathrm{Cl}(0,d) \otimes \mathrm{Cl}(2,0) \to \mathrm{Cl}(d+2,0)$ by its value on the generators Γ_a of $\mathrm{Cl}(d+2,0)$ as

$$\Gamma_a = \begin{cases} \Gamma'_a \otimes \Gamma''_* \,, \text{for } 1 \leq a \leq d \,, \\ \mathrm{id} \otimes \Gamma''_1 \,, \text{for } a = d+1 \,, \\ \mathrm{id} \otimes \Gamma''_2 \,, \text{for } a = d+2 \,. \end{cases} \tag{1.93}$$

Indeed, all distinct Γ_a anticommute because $\Gamma''_1, \Gamma''_2, \Gamma''_*$ anticommute. Moreover,

$$(\Gamma'_a \otimes \Gamma_*)^2 = (\Gamma'_a)^2 \otimes (\Gamma_*)^2 = \mathrm{id}' \otimes (-\,\mathrm{id}'') = -\,\mathrm{id}\,, \tag{1.94}$$

$$(\mathrm{id} \otimes \Gamma''_1)^2 = (\mathrm{id})^2 \otimes (\Gamma''_1)^2 = -\,\mathrm{id}\,, \tag{1.95}$$

$$(\mathrm{id} \otimes \Gamma''_2)^2 = (\mathrm{id})^2 \otimes (\Gamma''_2)^2 = -\,\mathrm{id}\;. \tag{1.96}$$

Checking 2. is entirely the same.

We check 3. Let $\Gamma'_1 \ldots, \Gamma'_p, \tilde{\Gamma}'_1, \ldots, \tilde{\Gamma}'_q$ be the generators of $\mathrm{Cl}(p,q)$ and $\Gamma''_1, \tilde{\Gamma}''_1$ be the generators of $\mathrm{Cl}(1,1)$. Define a map $\mathrm{Cl}(p,q) \otimes \mathrm{Cl}(1,1) \to \mathrm{Cl}(p+1,q+1)$ by its value on the generators $\Gamma_a, \tilde{\Gamma}_a$ of $\mathrm{Cl}(p+1,q+1)$ as

$$\Gamma_a = \begin{cases} \Gamma'_a \otimes \Gamma''_* \,, \text{for } 1 \leq a \leq p \,, \\ \mathrm{id} \otimes \Gamma''_1 \,, \text{for } a = p+1 \,. \end{cases} \tag{1.97}$$

$$\tilde{\Gamma}_a = \begin{cases} \tilde{\Gamma}'_a \otimes \Gamma''_* \,, \text{for } 1 \leq a \leq p \,, \\ \mathrm{id} \otimes \tilde{\Gamma}''_1 \,, \text{for } a = p+1 \,. \end{cases} \tag{1.98}$$

As before, all distinct $\Gamma_a, \tilde{\Gamma}_a$ anticommute since $\Gamma''_1, \tilde{\Gamma_1}'', \Gamma''_*$ anticommute. Moreover, $(\Gamma''_*)^2 = \mathrm{id}''$ so that $(\Gamma_a)^2 = -\,\mathrm{id}$ and $(\tilde{\Gamma}_a)^2 = \mathrm{id}$ as desired. □

Finally, before we classify Clifford algebras by constructing them from the smaller ones, we need the following lemma.

Lemma 1.23. *The following are isomorphic real algebras:*

1. $\mathrm{Mat}_m(\mathbb{R}) \otimes \mathrm{Mat}_n(\mathbb{R}) \cong \mathrm{Mat}_{nm}(\mathbb{R})$,

2. $\mathrm{Mat}_n(\mathbb{R}) \otimes_{\mathbb{R}} K \cong \mathrm{Mat}_n(K)$, *for* $K = \mathbb{C}, \mathbb{H}$ *and* $n \geq 1$,

3. $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$,

4. $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathrm{Mat}_2(\mathbb{C})$,

5. $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathrm{Mat}_4(\mathbb{R})$.

Proof. We check 1. This can be done using the Kronecker product and using standard matrix bases.

We check 2. This is immediate for $K = \mathbb{C}$. The non-commutative case of $\mathbb{H}$ requires one to define an ordering on $\mathbb{H}$ multiplication, but aside from that is immediate. Indeed, in both cases the standard basis e_{ab} of $\mathrm{Mat}_n(\mathbb{R})$ simply now can have coefficients in K.

We check 3. This follows from defining a $\mathbb{R}$-linear map $\mathbb{C} \oplus \mathbb{C} \to \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ by

$$(1,0) \mapsto \frac{1}{2}(1 \otimes 1 + i \otimes i)\,, \qquad (0,1) \mapsto \frac{1}{2}(1 \otimes 1 - i \otimes i)\,, \tag{1.99}$$

$$(i,0) \mapsto \frac{1}{2}(i \otimes 1 - 1 \otimes i)\,, \qquad (0,i) \mapsto \frac{1}{2}(i \otimes 1 + 1 \otimes i)\,. \tag{1.100}$$

This maps four basis vectors to four basis vectors and preserves products, so it is an isomorphism.

We check 4. We can write any $x \in \mathbb{H}$ in terms of two complex numbers a, b via $(ij = k)$

$$x = a + bj = a_1 + a_2 i + b_1 j + b_2 k\,. \tag{1.101}$$

Hence $\mathbb{H} \cong \mathbb{C}^2$ as a complex vector space and $\mathrm{End}_{\mathbb{C}}(\mathbb{H}) \cong \mathrm{End}_{\mathbb{C}}(\mathbb{C}^2) \cong \mathrm{Mat}_2(\mathbb{C})$. Define, by universal property, a $\mathbb{R}$-linear map $\pi : \mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \to \mathrm{End}_{\mathbb{C}}(\mathbb{H})$ via

$$\pi(z,x)(y) := zx\bar{y}\,, \tag{1.102}$$

where $\bar{i} = -i$, $\bar{j} = -j$, $\bar{k} = -k$ and $\overline{xy} = \bar{y}\bar{x}$ is conjugation. This is an algebra homomorphism since

$$\pi(z,x)\pi(z',x')y = zz'y\bar{x'}\bar{x} = zz'y\overline{xx'} = \pi(zz', xx')y\,. \tag{1.103}$$

Moreover, we can check this maps a $\mathbb{R}$-basis to a $\mathbb{R}$-basis.

We check 5. For the last isomorphism we play the same game but instead with $\mathbb{H} \cong \mathbb{R}^4$ and $\mathrm{End}_{\mathbb{R}}(\mathbb{H}) \cong \mathrm{End}_{\mathbb{R}}(\mathbb{R}^4) \cong \mathrm{Mat}_4(\mathbb{R})$. That is, our map is now $\pi : \mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \to \mathrm{End}_{\mathbb{R}}(\mathbb{H})$ with

$$\pi(x,y)(w) := xw\bar{y}\,. \tag{1.104}$$

□

Theorem 1.24 (Cartan/Bott). *The real Clifford algebras are classified and this classification exhibits 8-periodic behaviour in $p - q$.*

$p - q \pmod 8$	$\mathrm{Cl}(p,q)$	N
$0, 6$	$\mathrm{Mat}_N(\mathbb{R})$	$2^{d/2}$
$2, 4$	$\mathrm{Mat}_N(\mathbb{H})$	$2^{(d-2)/2}$
$1, 5$	$\mathrm{Mat}_N(\mathbb{C})$	$2^{(d-1)/2}$
3	$\mathrm{Mat}_N(\mathbb{H}) \oplus \mathrm{Mat}_N(\mathbb{H})$	$2^{(d-3)/2}$
7	$\mathrm{Mat}_N(\mathbb{R}) \oplus \mathrm{Mat}_N(\mathbb{R})$	$2^{(d-1)/2}$

Proof. We will need to divide into cases of $p - q \pmod 8$. Our convention will be that $V^{\otimes n}$ is n-tensor products of the vector space V and that $n = 0$ corresponds to $V^{\otimes 0} \cong \mathbb{R}$. Intuitively, the lemmas allow us to construct *any* Clifford algebra $\mathrm{Cl}(p,q)$ from our low-dimensional examples via tensor products. Moreover, this ensures such all Clifford algebras will be a matrix algebra or direct sum of matrix algebras. Thus, the theorem is plausible. It remains to classify them in the exact sense as the table. First observe the useful facts (which follow from the last two lemmas)

$$\mathrm{Cl}(8,0) \cong \mathrm{Cl}(0,8) \tag{1.105}$$

$$\cong \mathrm{Cl}(2,0) \otimes \mathrm{Cl}(2,0) \otimes \mathrm{Cl}(0,2) \otimes \mathrm{Cl}(0,2) \tag{1.106}$$

$$\cong \mathbb{H} \otimes \mathbb{H} \otimes \mathrm{Mat}_2 \mathbb{R} \otimes \tag{1.107}$$

$$\cong \mathrm{Mat}_{16}(\mathbb{R})\,, \tag{1.108}$$

and

$$\text{Cl}(n+8,0) \cong \text{Cl}(n,0) \otimes \text{Cl}(8,0)\,, \tag{1.109}$$
$$\text{Cl}(0,n+8) \cong \text{Cl}(0,n) \otimes \text{Cl}(0,8)\,, \tag{1.110}$$

and

$$\text{Cl}(1,1)^{\otimes n} \cong \text{Mat}_2(\mathbb{R})^{\otimes n} \cong \text{Mat}_{2^n}(\mathbb{R})\,. \tag{1.111}$$

The key point to realise here is that both $\text{Cl}(8,0), \text{Cl}(1,1)^{\otimes n}$ are real matrix algebras and, by the lemma, they will preserve any $\mathbb{R}, \mathbb{C}, \mathbb{H}$ they tensor product with. Thus, they 'trivially' change the structure of the algebra in the sense of the table.

Assume $p \geq q \geq 0$, then

$$\text{Cl}(p,q) \cong \text{Cl}(1,1)^{\otimes q} \otimes \text{Cl}(p-q,0) \cong \text{Mat}_{2^q}(\mathbb{R}) \otimes \text{Cl}(p-q,0) \tag{1.112}$$

Assume $q \geq p \geq 0$, then

$$\text{Cl}(p,q) \cong \text{Cl}(1,1)^{\otimes p} \otimes \text{Cl}(0,q-p) \cong \text{Mat}_{2^p}(\mathbb{R}) \otimes \text{Cl}(0,q-p) \tag{1.113}$$

Without loss of generality, we may also assume $p-q, q-p$ to be between 0 and 8 by using $\text{Cl}(8,0) \cong \text{Cl}(0,8) \cong \text{Mat}_{16}(\mathbb{R})$ and merely change the size of the real matrix algebra.

We

Thus, it suffices to classify $\text{Cl}(n,0), \text{Cl}(0,n)$ for $n \pmod 8$ to get everything. Moreover, we can already say that $0 \cong p-q \pmod 8$ correspond to $\text{Mat}_N(\mathbb{R})$ due to our knowledge of $\text{Cl}(1,1), \text{Cl}(8,0), \text{Cl}(0,8)$.

The full list is then

$$\text{Cl}(1,0) \cong \mathbb{C}\,, \tag{1.114}$$
$$\text{Cl}(0,1) \cong \mathbb{R} \oplus \mathbb{R}\,, \tag{1.115}$$
$$\text{Cl}(2,0) \cong \mathbb{H}\,, \tag{1.116}$$
$$\text{Cl}(0,2) \cong \text{Mat}_2(\mathbb{R})\,, \tag{1.117}$$
$$\text{Cl}(3,0) \cong \text{Cl}(0,1) \otimes \text{Cl}(2,0) \cong \mathbb{H} \oplus \mathbb{H}\,, \tag{1.118}$$
$$\text{Cl}(0,3) \cong \text{Cl}(1,0) \otimes \text{Cl}(0,2) \cong \text{Mat}_2(\mathbb{C})\,, \tag{1.119}$$
$$\text{Cl}(4,0) \cong \text{Cl}(0,2) \otimes \text{Cl}(2,0) \cong \text{Mat}_2(\mathbb{H})\,, \tag{1.120}$$
$$\text{Cl}(0,4) \cong \text{Cl}(0,2) \otimes \text{Cl}(2,0) \cong \text{Mat}_2(\mathbb{H})\,, \tag{1.121}$$
$$\text{Cl}(5,0) \cong \text{Cl}(0,3) \otimes \text{Cl}(2,0) \cong \text{Mat}_4(\mathbb{C})\,, \tag{1.122}$$
$$\text{Cl}(0,5) \cong \text{Cl}(3,0) \otimes \text{Cl}(0,2) \cong \text{Mat}_2(\mathbb{H}) \oplus \text{Mat}_2(\mathbb{H})\,, \tag{1.123}$$
$$\text{Cl}(6,0) \cong \text{Cl}(0,4) \otimes \text{Cl}(2,0) \cong \text{Mat}_8(\mathbb{R})\,, \tag{1.124}$$
$$\text{Cl}(0,6) \cong \text{Cl}(4,0) \otimes \text{Cl}(0,2) \cong \text{Mat}_4(\mathbb{H})\,, \tag{1.125}$$
$$\text{Cl}(7,0) \cong \text{Cl}(0,5) \otimes \text{Cl}(2,0) \cong \text{Mat}_8(\mathbb{R}) \oplus \text{Mat}_8(\mathbb{R})\,, \tag{1.126}$$
$$\text{Cl}(0,7) \cong \text{Cl}(5,0) \otimes \text{Cl}(0,2) \cong \text{Mat}_8(\mathbb{C})\,, \tag{1.127}$$
$$\text{Cl}(8,0) \cong \text{Mat}_{16}(\mathbb{R})\,, \tag{1.128}$$
$$\text{Cl}(0,8) \cong \text{Mat}_{16}(\mathbb{R})\,. \tag{1.129}$$

Hence, $1,5 \equiv p-q \pmod 8$ give $\text{Mat}_N(\mathbb{C})$. Next, $7 \equiv p-q \pmod 8$ gives $\text{Mat}_N(\mathbb{R}) \oplus \text{Mat}_N(\mathbb{R})$. Next, $2,4 \equiv p-q \pmod 8$ give $\text{Mat}_N(\mathbb{H})$. Next, $0,6 \equiv p-q \pmod 8$ gives $\text{Mat}_N(\mathbb{R})$. Next, $3 \equiv p-q \pmod 8$ gives $\text{Mat}_N(\mathbb{H}) \oplus \text{Mat}_N(\mathbb{H})$. □

To complete this subsection we also mention the classification of the even subalgebras $\text{Cl}(p,q)^0$. This is quite easy because of the following result.

Lemma 1.25. *The following isomorphisms hold:*

1. $\mathrm{Cl}(p,q)^0 \cong \mathrm{Cl}(p-1,q)$ *for* $p \geq 1$.
2. $\mathrm{Cl}(p,q)^0 \cong \mathrm{Cl}(q-1,p)$ *for* $q \geq 1$.

Proof. We check 1. If Γ'_a generate $\mathrm{Cl}(p,q)$, define $\Gamma_a := \Gamma'_{a+1}\Gamma'_1$ for $2 \leq a \leq d = p+q$. Notice $(\Gamma'_1)^2 = -1$. We can check that

$$\Gamma_a \Gamma_b = \Gamma'_{a+1}\Gamma'_1\Gamma_{b+1}\Gamma'_1 = \Gamma'_{a+1}\Gamma'_{b+1} \tag{1.130}$$

gives the product of any two Γ'_a, Γ'_b with $a, b \geq 2$. Hence, Γ_a indeed generate the even subalgebra $\mathrm{Cl}(p,q)^0$. Moreover, they still maintain their anticommuting nature and square to $-1, 1$ in the ways needed for them to generate $\mathrm{Cl}(p-1,q)$.

We check 2. The same proof almost holds, but now we use Γ'_d with $(\Gamma'_d)^2 = 1$ instead of Γ'_1. This amounts to flipping the sign of the squares of Γ_a. Hence, we generate $\mathrm{Cl}(q-1,p)$ as stated. □

Theorem 1.26. *The even parts of real Clifford algebras are classified and this classification exhibits 8-periodic behaviour in* $p-q$.

$p-q \pmod 8$	$\mathrm{Cl}(p,q)^0$	N
$1, 7$	$\mathrm{Mat}_N(\mathbb{R})$	$2^{(d-1)/2}$
$3, 5$	$\mathrm{Mat}_N(\mathbb{H})$	$2^{(d-3)/2}$
$2, 6$	$\mathrm{Mat}_N(\mathbb{C})$	$2^{(d-2)/2}$
4	$\mathrm{Mat}_N(\mathbb{H}) \oplus \mathrm{Mat}_N(\mathbb{H})$	$2^{(d-4)/2}$
0	$\mathrm{Mat}_N(\mathbb{R}) \oplus \mathrm{Mat}_N(\mathbb{R})$	$2^{(d-2)/2}$

1.5 Classification of representations of real Clifford algebras

As a result of classification of Clifford algebras, their reps amount to reps of matrix algebras $\mathrm{Mat}_N(\mathbb{R})$, $\mathrm{Mat}_N(\mathbb{C})$, $\mathrm{Mat}_N(\mathbb{H})$. We use some general results from ring theory to classify such reps. In particular, we will find that a matrix algebra $\mathrm{Mat}_N(D)$ with D a $\mathbb{R}$-division algebra has one real irrep given by its standard action on column vectors D^N. There is still the trivial rep, but we mostly ignore this.

Moreover, such an algebra is *semisimple* and thus every real rep V decomposes into a direct sum of real irreps, which, in this case, amounts to various copies of D^N and the trivial rep. Remember a finite-dimensional $\mathbb{R}$-algebra (like D) is just a $\mathbb{R}$-vector space with some multiplication, hence $D^N \cong \mathbb{R}^{\dim(D)N}$ as real vector spaces.

We give some definitions/results from ring theory relating to this. By ring we always mean unital associative ring (in particular the non-associative octonions $\mathbb{O}$ are not a ring in our conventions).

First, the ring definitions.

Definition 1.27. A ring R is *simple* if its two-sided ideals are trivial i.e. only 0 and R are two-sided ideals.

Definition 1.28. A ring R is *left (right) Artinian* if it does not have an infinite descending chain of left (right) ideals. It is called *Artinian* if both left/right sides hold.

Second, the module definitions.

Definition 1.29. A left (right) R-module M is an abelian group equipped with left (right) 'scalar multiplication' by R obeying similar axioms to vector spaces.

Definition 1.30. An R-module M is *simple* if its submodules are trivial, i.e. 0 or M.

Definition 1.31. An R-module is *semisimple* if it is the direct sum of simple R-modules. In particular, if it is the direct sum of simple/minimal ideals.

Definition 1.32. A ring R is *semisimple* if it is a semisimple module over itself (left-semisimple and right-semisimple coincide).

Associative algebras are special cases of rings in the sense that they are also vector spaces over some field k. In particular, for an associative algebra A an A-module is the same thing as a rep of A over its field k. This is precisely what we will use convert our modules into real reps.

The following result clarifies semisimple rings better.

Proposition 1.33. *R is a semisimple ring iff every R-module M is semisimple.*

Proof. One direction is trivial. Assume R is semisimple. Let $m \in M$ be non-zero and define a R-module homomorphism $\phi : R \to M$ by $r \mapsto rm$. Since R is semisimple, then $Rm \cong R/\ker\phi$ is semisimple. Indeed, for some simple ideals I_α

$$Rm \cong R/\ker\phi \cong \left(\bigoplus_\alpha I_\alpha\right)/\ker\phi \tag{1.131}$$

but $\ker\phi \subseteq R$ itself is an ideal and hence must be a direct sum of some of the simple ideals I_α. Hence,

$$rm \cong \left(\bigoplus_\alpha I_\alpha\right)/\ker\phi \cong \bigoplus_{\alpha_\beta} I_{\alpha_\beta} \tag{1.132}$$

for some subset $\{\alpha_\beta\} \subseteq \{\alpha\}$ of indices. We can continue more non-zero taking elements $m' \notin Rm$ and eventually reach a decomposition

$$M = \bigoplus_\gamma Rm_\gamma \tag{1.133}$$

for some subset $\{m_\gamma\} \subseteq M$. Hence M is semisimple. □

We take the following results for granted.

Proposition 1.34. *The following are true:*

1. *A simple ring R is semisimple iff R is left Artinian (see this post).*
2. *A simple ring R is semisimple iff R is right Artinian (see this post).*
3. *If R is a simple Artinian ring, then it has one simple left (right) R-module up to isomorphism (see this post).*
4. *Let D be a division ring, then the ring $\mathrm{Mat}_N(D)$ is a simple Artinian ring (indeed these are in some sense the building blocks of semisimple Artinian rings used in the Artin-Wedderburn theorem).*

Clearly, the space of column vectors D^N gives a simple $\mathrm{Mat}_N(D)$-module since we can pick a matrix to take a given column vector to any other (leverage invertibility of elements of D). Hence it is cyclic and non zero, so it must be simple.

In the direct sum/product case $R = \mathrm{Mat}_N(D) \times \mathrm{Mat}_{N'}(D')$[8] both D^N and $D'^{N'}$ are simple R-modules by letting one of the rings act as the identity. We can check that there are no other simple modules because any $R \times S$ module M canonically decomposes as $M = M_R \oplus M_S$, where $M_R := (1,0)M$ and $M_S := (0,1)M$. This decomposition leads to the following.

[8] Really one should only ever consider direct products of rings because there is are natural projection map $(1,1) \mapsto 1$ of $R \times S \to R, S$ with coincides with the universal property of products. Similar considerations with algebras. The universal property of coproducts (direct sums) is the exact opposite and one needs natural inclusion morphisms $R, S \to R \sqcup S$ which don't exist (for unital rings/algebras). With vector spaces/modules products and coproducts coincide.

Proposition 1.35. *Let $M = M_R \oplus M_S$ be any $R \times S$ module decomposed in the canonical way. Then M is simple iff M_R is a simple R-module and $M_S = 0$ or M_S is a simple S-module and $M_R = 0$.*

Proof. Assume M is simple and $M_R \neq 0$. Observe that

$$(r, s) \cdot (m_r, m_s) = ((r, s) \cdot m_r, (r, s) \cdot m_s) = (rm_r, sm_s) \tag{1.134}$$

separates the $R \times S$ action. Simplicity requires that any two vectors $m = (m_r, m_s)$, $m' = (m'_r, m'_s)$ are related by the action of some (r, s). In particular, this means that both M_R and M_S are cyclic. However, we find $M_S = 0$ because the non-zero vector $m = (m_r, 0)$ can must generate $M = M_R \oplus M_S$. Hence, $M_S = 0$. □

Moreover, the product $\mathrm{Mat}_N(D) \times \mathrm{Mat}_{N'}(D')$ is indeed semisimple because we can check the following result.

Proposition 1.36. *If R and S are semisimple, then $R \times S$ is semisimple (this holds for infinite products too).*

Proof. $R \times S$ acts naturally on itself as a module (i.e. on $R \oplus S \cong R \times S$ which are isomorphic as modules). Moreover, this module is already decomposed with $M_R = R, M_S = S$. Due to this, one can apply semisimplicity of R, S to show that $R \oplus S$ is a semisimple module. □

Thus, as a result we get the following.

Theorem 1.37. *The real Clifford algebras and their even subalgebras are classified, are semisimple and have exactly one or two simple modules (equivalently, real irreps). The tables below classifies them up to isomorphism.*

$p - q \pmod 8$	$\mathrm{Cl}(p, q)$	N	Modules
$0, 6$	$\mathrm{Mat}_N(\mathbb{R})$	$2^{d/2}$	$\mathbb{R}^N$
$2, 4$	$\mathrm{Mat}_N(\mathbb{H})$	$2^{(d-2)/2}$	$\mathbb{H}^N$
$1, 5$	$\mathrm{Mat}_N(\mathbb{C})$	$2^{(d-1)/2}$	$\mathbb{C}^N$
3	$\mathrm{Mat}_N(\mathbb{H}) \oplus \mathrm{Mat}_N(\mathbb{H})$	$2^{(d-3)/2}$	$\mathbb{H}^N_1, \mathbb{H}^N_2$
7	$\mathrm{Mat}_N(\mathbb{R}) \oplus \mathrm{Mat}_N(\mathbb{R})$	$2^{(d-1)/2}$	$\mathbb{R}^N_1, \mathbb{R}^N_2$
$p - q \pmod 8$	$\mathrm{Cl}(p, q)^0$	N	Modules
$1, 7$	$\mathrm{Mat}_N(\mathbb{R})$	$2^{(d-1)/2}$	$\mathbb{R}^N$
$3, 5$	$\mathrm{Mat}_N(\mathbb{H})$	$2^{(d-3)/2}$	$\mathbb{H}^N$
$2, 6$	$\mathrm{Mat}_N(\mathbb{C})$	$2^{(d-2)/2}$	$\mathbb{C}^N$
4	$\mathrm{Mat}_N(\mathbb{H}) \oplus \mathrm{Mat}_N(\mathbb{H})$	$2^{(d-4)/2}$	$\mathbb{H}^N_1, \mathbb{H}^N_2$
0	$\mathrm{Mat}_N(\mathbb{R}) \oplus \mathrm{Mat}_N(\mathbb{R})$	$2^{(d-2)/2}$	$\mathbb{R}^N_1, \mathbb{R}^N_2$

Moreover, the even parts of Clifford algebras are signature-independent*, that is $\mathrm{Cl}(p, q)^0 \cong \mathrm{Cl}(q, p)^0$.*

Definition 1.38. As eluded to in the brief discussion of complex semisimple Lie algebras, we name the irreps as follows.

1. The irreps of $\mathrm{Cl}(p, q)$ are called *pinor* irreps.

2. The irreps of $\mathrm{Cl}(p, q)^0$ are called *spinor* irreps.

The reason for the naming is that they are irreps of the pin and spin groups respectively. Moreover, these representations are quite special because they are *not* representations of the orthogonal groups. The following theorem clarifies the situtation.

Theorem 1.39. *The following hold:*

1. *The pinor and spinor irreps are irreps of* $\mathrm{Pin}(p,q)$ *and* $\mathrm{Spin}(p,q)$ *respectively.*
2. *The spinor irreps are also irreps of* $\mathrm{Spin}^+(p,q)$*.*
3. *The pinor and spinor irreps are not representations of* $SO^+(p,q)$ *(and hence not of the other orthogonal groups).*
4. *The previously described map* $\mathfrak{so}(p,q) \to \mathrm{Cl}(p,q)$ *is natural in the sense that the exponential (defined as a formal power series) maps into* $\mathrm{Spin}^+(p,q)$*.*
5. *When the spinor irreps are complexified, they are what we previously called the fundamental spinor irreps of* $\mathfrak{so}(d,\mathbb{C})$ *(i.e. the reps corresponding to fundamental weights of* $\mathfrak{so}(d,\mathbb{C})$*).*

Proof. We check 1.

Fix a basis e_a of V and hence also the standard one for $\mathrm{Cl}(p,q)$. It is easy to see that the pin group contains any product of basis vectors e_a and that such products span $\mathrm{Cl}(p,q)$. It is also obvious that a rep of $\mathrm{Cl}(p,q)$ restricts to a rep of $\mathrm{Pin}(p,q)$. Therefore, if a rep of $\mathrm{Cl}(p,q)$ is cyclic (an irrep), it must remain cyclic when restricted to $\mathrm{Pin}(p,q)$. The same argument readily goes through for $\mathrm{Cl}(p,q)^0$ and $\mathrm{Spin}(p,q)$.

We check 2.

I would like to give a direct proof of this... Alternatively, we argue through Lie algebras using the other results. The spinor irreps are irreps of $\mathfrak{so}(p,q)$ by 5. and using complexification/real structure. Hence, by 4., through the exponential map they are reps of the connected Lie group $\mathrm{Spin}^+(p,q)$. From standard results, the rep is irreducible for the Lie algebra iff it is irreducible for the connected Lie group under consideration.

One can realise this quite concretely. Denote the group rep in terms of the map ρ, then $\rho(e^x) = e^{\rho_*(x)}$ by basic differential geometry, where the differential ρ_* coincides with the rep of $\mathfrak{so}(p,q)$ originally given. Being an irrep of $\mathfrak{so}(p,q)$ means there is no proper subspace invariant under $\rho_*(x)$. Hence, there is no subspace invariant under $\rho(e^x)$ and, by connectedness of $\mathrm{Spin}^+(p,q)$, every group element is expressible as a product $g = e^{x_1} \dots e^{x_r}$, so it must be an irrep of the group too.

We check 3.

This follows from $\mathrm{Cl}(p,q)$ being faithfully represented on pinor irreps. Indeed, due to this $\mathrm{Pin}(p,q)$ is faithfully represented. Hence the natural map $\widetilde{\mathrm{Ad}}$ which sends $\pm x \mapsto \widetilde{\mathrm{Ad}}(x)$ only allows one to treat a pinor representation as a *projective representation* of $O(p,q)$. That is $\Lambda \in O(p,q)$ acts up to sign on the pinor irrep. Because the double covering is not trivial, i.e. $\mathrm{Pin}(p,q)$ is not a trivial fibre bundle, then there is no way to decide the sign of which to act $\Lambda \in O(p,q)$ on a pinor irrep, hence it can never be a rep of $O(p,q)$. Similar things can be said for spinor irreps.

We check 4.

Let $x \in \mathfrak{so}(p,q) \subseteq \mathrm{Cl}(p,q)^0$. Then due to the way we embed $\mathfrak{so}(p,q)$ into the Clifford algebra $x = x^{ab} e_{ab}$ where $e_{ab} = \frac{1}{2}(e_a e_b - e_b e_a)$ with $a < b$ for some basis e_a of V. From here we recall that $y \in \mathrm{Spin}^+(p,q)$ iff y is even, invertible and $N(y) = 1$. Consider the formal power series e^x. Clearly, it is even since x is even, it is invertible with inverse e^{-x} and

$$N(e^x) = e^x \kappa(e^x) = e^x e^{-x} = 1\,. \tag{1.135}$$

Hence, if it converges, then $e^x \in \mathrm{Spin}^+(p,q)$.

We check 5.

This is proved inside Fulton and Harris section 19.2. In particular, it amounts to checking the weights of the complexified reps when it is viewed as a rep of $\mathfrak{so}(d,\mathbb{C})$. □

The way to realise all of these as real irreps in the traditional sense of having an action on a real vector space is simply to replace $\mathbb{C}^N \cong \mathbb{R}^{2N}$ and $\mathbb{H}^N \cong \mathbb{R}^{4N}$ as real vector spaces. However, $\mathrm{Mat}_N(\mathbb{C})$ acting on $\mathbb{C}^N$ still has another natural structure within it, namely that it is complex linear. In particular, this means endowing a real vector space *complex structure*.

Definition 1.40 (Complex structure)**.** A *complex structure* on a real vector space V is a real linear map $I : V \to V$ such that $I^2 = -\,\mathrm{id}$, i.e. it acts like multiplication by $i \in \mathbb{C}$.

Indeed, a complex structure on V makes V into a complex vector space by defining

$$zv := av + bI(v)\,, \tag{1.136}$$

where $z = a + bi$, and one can readily check that $(zw)v = z(wv)$. The complex structure also gives a real vector space decomposition $V = V_1 \oplus V_2$ via

$$v = \frac{1}{2}(v - I(v)) + \frac{1}{2}(v + I(v)) = v_1 + v_2\,, \tag{1.137}$$

where $I(v_1) = v_2$ and $I(v_2) = -v_1$. Choosing a basis of $\{e_1, \dots, e_N\}$ of V_1 allows us to write I as the matrix

$$I = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}. \tag{1.138}$$

When V is instead viewed as a complex vector space, we instead find that $I = i\,\mathrm{id}$ is just multiplication by i.

As it is useful, we also reverse this procedure here, i.e. start with $\mathbb{C}^N$ and describe $\mathbb{R}^{2N}$ in terms of it. Indeed, pick the standard complex basis e_a of $\mathbb{C}^N$ and define a real basis as (e_a, ie_a). We can readily identify this with the standard basis of $\mathbb{R}^{2N}$. Then define the complex structure $I : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ by

$$I(e_a) = ie_a\,, \qquad I(ie_a) = -e_a\,, \qquad I = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}. \tag{1.139}$$

At the level of matrices, we have a $\mathbb{R}$-algebra homomorphism $\mathrm{Mat}_N(\mathbb{C}) \to \mathrm{Mat}_{2N}(\mathbb{R})$ described by

$$A \mapsto A_{\mathbb{R}} = \begin{pmatrix} \mathrm{Re}(A) & -\mathrm{Im}(A) \\ \mathrm{Im}(A) & \mathrm{Re}(A) \end{pmatrix}. \tag{1.140}$$

Indeed, one can check this is consistent with

$$Ae_a = A_a{}^b e_b = \mathrm{Re}(A)_a{}^b e_b + \mathrm{Im}(A)_a{}^b (ie_b)\,, \tag{1.141}$$

$$A(ie_a) = iA_a{}^b e_b = \mathrm{Re}(A)_a{}^b (ie_b) - \mathrm{Im}(A)_a{}^b e_b\,. \tag{1.142}$$

It is also instructive to realise that the image of the map $\mathrm{Mat}_N(\mathbb{C}) \to \mathrm{Mat}_{2N}(\mathbb{R})$ is precisely those matrices $A_{\mathbb{R}}$ such that $A_{\mathbb{R}} I = I A_{\mathbb{R}}$, that is 'complex linearity' still holds, so that our Clifford algebra still maintains this property.

In the quaternionic case $\mathrm{Mat}_N(\mathbb{H})$ a similar thing happens. However, it is helpful to first step from $\mathbb{H}$ to $\mathbb{C}$ and work with a complex vector space with a *quaternionic structure*. In particular, $\mathbb{H}^N \cong \mathbb{C}^{2N}$ as complex vector spaces.

Definition 1.41 ($\mathbb{C}$-quaternionic structure)**.** A *quaternionic structure* on a complex vector space V is a conjugate-linear map $J : V \to \bar{V}$ such that $J^2 = -\,\mathrm{id}$.[9]

[9] Here $\bar{V}$ denotes the *complex conjugate* of the complex vector space V. As sets $V = \bar{V}$ and the only difference is that the scalar multiplication is now $z \cdot v = \bar{z}v$. Moreover, $\bar{V}$ is still a complex vector space itself.

We first introduce the following notation for decomposing quaternions into complex numbers

$$a + bi + cj + dk = (a + bi) + (c + di)j\,, \tag{1.143}$$
$$\mathrm{Re}_{\mathbb{C}}(a + bi + cj + dk) = a + bi\,, \tag{1.144}$$
$$\mathrm{Im}_{\mathbb{C}}(a + bi + cj + dk) = c + di\,. \tag{1.145}$$

Indeed, a quaternionic structure on V makes V into a left $\mathbb{H}$-module by defining

$$qv := zv + wJ(v)\,, \tag{1.146}$$

where $q = z + wj$, and one can check that $(qq')v = q(q'v)$. By using conjugation, we can make it into a right-$\mathbb{H}$ module

$$vq := \bar{z} - wJ(v)\,, \tag{1.147}$$

where one should think roughly that $vq \equiv v\bar{q} \equiv v(\bar{z} - wj)$. One can check that $v(qq') = (vq)q'$. This didn't matter in the case of complex structures because $\mathbb{C}$ is commutative, but left/right actions are important since $\mathbb{H}$ is not. The quaternionic structure also gives a complex vector space decomposition $V = V_1 \oplus V_2$ via

$$v = \frac{1}{2}(v - J(v)) + \frac{1}{2}(v + J(v)) = v_1 + v_2\,, \tag{1.148}$$

where $J(v_1) = v_2$ and $J(v_2) = -v_1$. Choosing a basis of $\{e_1, \ldots, e_N\}$ of V_1 allows us to get a basis $(e_a, J(e_a))$ of V, with the property

$$J(e_a) = J(e_a)\,, \qquad J(J(e_a)) = -e_a\,. \tag{1.149}$$

Note that J is *not* represented by a complex matrix because it is conjugate-linear. When V is instead viewed as a left $\mathbb{H}$-module we instead find that $J = j\,\mathrm{id}$ is just multiplication by j. Similarly, when V is viewed as a right $\mathbb{H}$-module we find that $J = -j\,\mathrm{id}$.

We again reverse this procedure, i.e start with $\mathbb{H}^N$ and describe $\mathbb{C}^{2N}$ in terms of it. Indeed, pick the standard quaternionic basis e_a of $\mathbb{H}^N$ and and define a complex basis as (e_a, je_a). We can readily identify this with the standard basis of $\mathbb{C}^{2N}$. Define the quaternionic structure $J : \mathbb{C}^{2N} \to \mathbb{C}^{2N}$ by

$$J(e_a) = je_a\,, \qquad J(je_a) = -e_a\,, \qquad J = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}\,. \tag{1.150}$$

The matrix can only show its values on basis vectors and any linear combinations must be interpreted using complex conjugation. We keep J's 'complex matrix' here merely for comparison with a complex structure on a real vector space discussed above. Schematically, a vector $v \in \mathbb{C}^{2N}$ obeys[10]

$$J(v) = J((v_1)^a e_a + (v_2)^a (je_a)) = (\bar{v}_1)^a (je_a) - (\bar{v}_2)^a e_a \equiv jv\,. \tag{1.151}$$

At the level of matrices, we have a $\mathbb{C}$-algebra homomorphism $\mathrm{Mat}_N(\mathbb{H}) \to \mathrm{Mat}_{2N}(\mathbb{C})$ described by

$$A \mapsto A_{\mathbb{C}} = \begin{pmatrix} \mathrm{Re}_{\mathbb{C}}(A) & -\mathrm{Im}_{\mathbb{C}}(A) \\ \mathrm{Im}_{\mathbb{C}}(A) & \mathrm{Re}_{\mathbb{C}}(A) \end{pmatrix}\,. \tag{1.152}$$

Indeed, one can check this is consistent with

$$Ae_a = A_a{}^b e_b = \mathrm{Re}_{\mathbb{C}}(A)_a{}^b e_b + \mathrm{Im}_{\mathbb{C}}(A)_a{}^b (je_b)\,, \tag{1.153}$$
$$A(je_a) = iA_a{}^b e_b = \mathrm{Re}_{\mathbb{C}}(A)_a{}^b (je_b) - \mathrm{Im}_{\mathbb{C}}(A)_a{}^b e_b\,. \tag{1.154}$$

[10] We need the complex conjugates here because $jz = \bar{z}j$ for a complex number z.

Moreover, the matrices $A_{\mathbb{C}}$ are precisely those which obey $A_{\mathbb{C}} J = J A_{\mathbb{C}} : V \to \bar{V}$, that is 'quaternionic linearity' is preserved on $\mathbb{C}^{2N}$.

Finally, to complete the story we simply take the complex basis (e_a, je_a) and make it into the real basis $((e_a, je_a), i(e_a, je_a)) = (e_a, je_a, ie_a, ke_a)$ and put a complex structure I on it, then $\mathbb{R}$-linearly map $\mathrm{Mat}_{2N}(\mathbb{C}) \to \mathrm{Mat}_{4N}(\mathbb{R})$. We find

$$J \mapsto J_{\mathbb{R}} = \begin{pmatrix} \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix} \end{pmatrix}, \qquad I = \begin{pmatrix} 0 & \begin{pmatrix} -I_N & 0 \\ 0 & -I_N \end{pmatrix} \\ \begin{pmatrix} I_N & 0 \\ 0 & I_N \end{pmatrix} & 0 \end{pmatrix} \tag{1.155}$$

These matrices obey the usual relations for $i, j, k \in \mathbb{H}$. Indeed,

$$I^2 = (J_{\mathbb{R}})^2 = -\,\mathrm{id}\,, \qquad I J_{\mathbb{R}} = -J_{\mathbb{R}} I\,. \tag{1.156}$$

Which implies the rest of the relations. Moreover, the $\mathbb{R}$-linear map $\mathrm{Mat}_N(\mathbb{H}) \to \mathrm{Mat}_{4N}(\mathbb{R})$ consists precisely of those matrices which commute with $I, J_{\mathbb{R}}$. Hence, we get the following definition.

Definition 1.42 ($\mathbb{R}$-quaternionic structure)**.** A *quaternionic structure* on a real vector space V is two $\mathbb{R}$-linear maps $I, J : V \to V$ such that $I^2 = J^2 = -\,\mathrm{id}$ and $IJ = -JI$.

Thus, we see that real irreps of Clifford algebras come alongside complex/quaternionic structures in general. This motivates the following definition.

Definition 1.43 (Complex and $\mathbb{R}$-quaternionic types)**.** We give complex and quaternionic cases.

1. Let A be an $\mathbb{R}$-algebra (G a group). A rep of A (G) on a real vector space V is of *complex type* if V has a complex structure I and A-actions (G-actions) commute with I.

2. Let A be an $\mathbb{R}$-algebra (G a group). A rep of A (G) on a real vector space V is of *quaternionic type* if V has a quaternionic structure I, J and A-actions (G-actions) commute with I, J.

Finally, we comment that a rep of $\mathrm{Cl}(p, q)$ on $\mathbb{C}^N$ or $\mathbb{H}^N \cong \mathbb{C}^{2N}$ is, of course, also a *complex rep* of $\mathrm{Cl}(p, q)$, i.e that $\mathrm{Cl}(p, q)$ is being represented on a complex vector space. In fact it will usually be easier to work directly with $\mathbb{C}$-vector spaces in the oncoming sections. So we will quickly develop this theory now.

We also have the simple, but extremely useful fact (almost identical to its special case, the Lie algebra version).

Proposition 1.44. *Let A be a $\mathbb{R}$-algebra, then any complex rep V gives a complex rep of its complexification $A_{\mathbb{C}}$. Moreover, by restriction we can take a complex rep of $A_{\mathbb{C}}$ to one for A. These operations are inverse.*

Proof. Given a complex rep $\rho : A \to \mathrm{End}(V)$ we need only define

$$\rho_{\mathbb{C}}(x + iy) := \rho(x) + i\rho(y) \tag{1.157}$$

for each $x, y \in A$ to lift it to a rep of $A_{\mathbb{C}}$ because ρ is $\mathbb{R}$-linear and already preserves A's multiplication and

$$(x + iy) \cdot_{\mathbb{C}} (z + iw) := xz - yw + i(xw + yz)\,. \tag{1.158}$$

□

In particular, this means that our real reps valued in $\mathbb{C}^N$ or $\mathbb{H} \cong \mathbb{C}^{2N}$ can be thought of instead as complex reps of complex Clifford algebras $\mathbb{C}\mathrm{l}(p + q) \cong \mathrm{Cl}(p, q) \otimes \mathbb{C}$.

2 Structure of complex Clifford algebras $\mathbb{C}\mathrm{l}(d)$

An advantage to what we have done so far is that most of it readily extends to the complex case. Moreover, the following proposition allows us to further justify that all information about real representations of $\mathbb{R}$-algebras (groups G) can be obtained from their complexifications.

Proposition 2.1. *Let A be a $\mathbb{R}$-algebra (group G), then any real rep V of A (G) can be realised as a 'real structure' on a complex representation $V_{\mathbb{C}} = V \otimes \mathbb{C}$.*

Proof. We have not defined what we mean by real structure yet, but will after this proof. The canonical real structure on the complexified vector space $V_{\mathbb{C}} = V \otimes \mathbb{C}$ is given by identifying V as the real part. Equivalently, it is given by specifying a conjugate-linear map $J : V_{\mathbb{C}} \to \bar{V}_{\mathbb{C}}$ such that $J^2 = \mathrm{id}$ and V is the eigenspace with eigenvalue one.

The rep of course preserves the real structure, i.e when restricted to act on V it maps into V. This is precisely because V was originally a rep. Equivalently, J commutes with the action of the rep on $V_{\mathbb{C}}$. □

Definition 2.2 (Real structure)**.** A *real structure* on a complex vector space V is a conjugate-linear map $J : V \to \bar{V}$ such that $J^2 = \mathrm{id}$, i.e. J acts as complex conjugation.

The canonical example when $V = \mathbb{C}^N$ with standard basis e_a is such that $J(e_a) = e_a$. Extending by conjugate-linearity gives the maps

$$J(e_a) = e_a\,, \qquad J(ie_a) = -ie_a\,. \tag{2.1}$$

Hence, when $\mathbb{C}^N$ is identified as a real vector space with basis (e_a, ie_a) as before, we find

$$J_{\mathbb{R}} = \begin{pmatrix} I_N & 0 \\ 0 & -I_N \end{pmatrix}\,. \tag{2.2}$$

In general one should think of J as corresponding to one (of infinitely many) choices of a decomposition $V = V_{\mathbb{R}} \oplus iV_{\mathbb{R}}$ via

$$v = \frac{1}{2}(v + Jv) + \frac{1}{2}(v - Jv)\,. \tag{2.3}$$

This motivates the corresponding rep theory definition.

Definition 2.3 (Real and quaternionic types)**.** We give the complex rep version of the previous definition.

1. Let A be an $\mathbb{R}$-algebra (G a group). A rep of A (G) on a $\mathbb{C}$-vector space V is of *real type* if V has a real structure J and A-actions (G-actions) commute with J.
2. Let A be an $\mathbb{R}$-algebra (G a group). A rep of A (G) on a real vector space V is of *quaternionic type* if V has a quaternionic structure J and A-actions (G-actions) commute with J.

Notice that this definition *still works* for $\mathbb{C}$-algebras. This is merely a statement about their reps being complex and $\mathbb{R} \subseteq \mathbb{C}$ causes no issues because it is contained in $\mathbb{C}$.

Comment: When considering V as giving a rep, there is no reason, in general, for V to be of real, complex or quaternionic type. If it is, then we are imposing an extra condition on the rep because it needs to commute with the maps I or J or both.

2.1 Definitions and conventions

This remains almost exactly the same except that bilinear forms over $\mathbb{C}$ are always expressible as $\delta = \mathrm{diag}(1, \ldots, 1)$ over some basis e_a. Hence we only get Clifford algebras of the form $\mathbb{C}\mathrm{l}(d) = \mathbb{C}\mathrm{l}(d, 0)$ and other 'signature' Clifford algebras are isomorphic to this (just change the basis of the vector space appropriately). Again, we use the quotient of the tensor algebra to construct $\mathbb{C}\mathrm{l}(d)$.

In particular we will usually let $V \cong \mathbb{C}^d$ denote the underlying vector space of the complex Clifford algebra $\mathbb{C}\mathrm{l}(d)$. Like the real case we will then have $V \to \mathbb{C}\mathrm{l}(d)$ with the relation

$$vw + wv = -2\delta(v, w) \,. \tag{2.4}$$

2.2 Natural maps

The main involution α and main anti-automorphism τ are the same. Signature-dependent maps can be defined, but are sort of unnatural because every complex bilinear form can be written as $\delta = \mathrm{diag}(1, \ldots, 1)$ in some basis e_a, so we don't bother with these.

We still have that $\alpha(v) = -v$ decomposes $\mathbb{C}\mathrm{l}(d)$ into two parts

$$x = \frac{1}{2}(x + \alpha(x)) + \frac{1}{2}(x - \alpha(x)) \,, \tag{2.5}$$

$$\mathbb{C}\mathrm{l}(d) = \mathbb{C}\mathrm{l}(d)^0 \oplus \mathbb{C}\mathrm{l}(d)^1 \,, \tag{2.6}$$

where $\mathbb{C}\mathrm{l}(d)^0$ contains linear combinations of even products ($\alpha(x) = x$) and $\mathbb{C}\mathrm{l}(d)^1$ contains linear combinations of odd products ($\alpha(x) = -x$).

Moreover, Clifford multiplication still preserves this $\mathbb{Z}_2$-grading, i.e.

$$\mathbb{C}\mathrm{l}(d)^i \oplus \mathbb{C}\mathrm{l}(d)^j \subseteq \mathbb{C}\mathrm{l}(d)^{i+j} \,, \tag{2.7}$$

where $i + j$ is interpreted mod 2. In particular, the product of two evens is even, the product of two odds is even and the product of an even and an odd is odd.

2.3 Lipschitz–Clifford group and pin and spin groups

A lot of section remains the same because Cartan-Dieudonné still holds and every construction that we previously used still holds with minor adjustments (like replacing $\mathbb{R}^\times$ with $\mathbb{C}^\times$). There is however a more significant difference in the topology of the groups though. Essentially, $\pi_1(\mathrm{Spin}(d)) \cong \pi_1(\mathrm{Spin}(d, \mathbb{C}))$ have the same fundamental groups as their positive-definite real versions.

The twisted adjoint representation remains the same $\widetilde{\mathrm{Ad}}(x)(y) = \alpha(x) y x^{-1}$. The definition and equivalent redefinitions of the complex Clifford-Lipschitz group $\Gamma(d, \mathbb{C})$ remain the same. That is

Theorem 2.4. *The following definitions are all equivalent*

1. $\Gamma(d, \mathbb{C}) := \{x \in \mathbb{C}\mathrm{l}(d)^\times \mid \widetilde{\mathrm{Ad}}(x)(v) \in V \quad \text{for all } v \in V\}$.
2. $\Gamma(d, \mathbb{C}) := \{\lambda v_1 \ldots v_r \mid \delta(v_i) \neq 0 \,, \lambda \in \mathbb{C}^\times\}$.
3. $\Gamma(d, \mathbb{C}) := \{x \in \mathbb{C}\mathrm{l}(d)^\times \mid N(x) \in \mathbb{C}^\times \cdot \mathrm{id}\}$.

We also have the analogous exact sequence

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow \Gamma(d, \mathbb{C}) \xrightarrow{\widetilde{\mathrm{Ad}}} O(d, \mathbb{C}) \longrightarrow 1 \,. \tag{2.8}$$

Moreover, it is realised in the same way, i.e. we can go through the same process as in the real case and show $N(\Gamma(d, \mathbb{C})) \subseteq \ker \widetilde{Ad} = \mathbb{C}^\times \cdot \mathrm{id}$.

Definition 2.5. The *complex pin group* is defined as the group (all definitions are equivalent by Theorem 2.4)

$$\mathrm{Pin}(d,\mathbb{C}) := \{x \in \Gamma(d,\mathbb{C}) \mid N(x) = 1\}\,, \tag{2.9}$$
$$\mathrm{Pin}(d,\mathbb{C}) := \{x \in \mathbb{C}\mathrm{l}(d)^\times \mid N(x) = 1\}\,, \tag{2.10}$$
$$\mathrm{Pin}(d,\mathbb{C}) := \{\pm v_1 \dots v_r \mid \delta(v_i) = -1\}\,. \tag{2.11}$$

The *complex spin group* is defined as $\mathrm{Spin}(d,\mathbb{C}) := \mathrm{Pin}^0(d,\mathbb{C})$, i.e. the even part. Moreover, every element of $\mathrm{Pin}(d,\mathbb{C})$ must satisfy $N(x) = 1$ so there is no notion of an orthochronous spin group here (similar thing happens in the real case when considering positive/negative-definite forms).

General comments: If one copies the real case completely and tries to use the <u>incorrect</u> definition

$$\mathrm{Pin}(d,\mathbb{C}) = \{x \in \Gamma(d,\mathbb{C}) \mid N(x) = \pm 1\} \tag{2.12}$$
$$= \{\lambda v_1 \dots v_r \mid \lambda = \pm 1, \pm i\,,\ \delta(v_i) = -1\}\,, \tag{2.13}$$

then one gets a fourfold $\mathbb{Z}_4 \cong \{\pm 1, \pm i\}$ covering of $O(d,\mathbb{C})$ not a double covering. The problem is that any element of $\Gamma(d,\mathbb{C})$ can be written as a product of unit vectors (v such that $v^2 = -\delta(v) = 1$) multiplied by some $\lambda \in \mathbb{C}^\times$ and the condition $N(x) = \pm 1$ allows $\lambda = \pm 1, \pm i$ because 1 of course has two square roots ± 1, but, due to working with $\mathbb{C}$, now -1 also has two square roots. On the otherhand, requiring just $N(x) = 1$ prevents this.

It is important to comment that complex spin groups $\mathrm{Spin}(d,\mathbb{C})$ are *not* the so-called spin-c groups $\mathrm{Spin}^c(d)$. In fact one can consider groups $\mathrm{Spin}^c(p,q)$ which highlights their dependence upon the choice of signature (p,q) of a real bilinear form η. Indeed, one usually defines the $\mathrm{Spin}^c(p,q)$ groups as

$$\mathrm{Spin}^c(p,q) = (\mathrm{Spin}(p,q) \times U(1))/\mathbb{Z}_2 \tag{2.14}$$

where we quotient by $(x,u) \sim (-x,-u)$. Hence it obeys the exact sequence

$$1 \longrightarrow U(1) \longrightarrow \mathrm{Spin}^c(p,q) \xrightarrow{\widetilde{\mathrm{Ad}}} SO^+(p,q) \longrightarrow 1\,. \tag{2.15}$$

Next we find that the topological aspects of these groups do change a bit in the complex case. They closely resemble the topology of the positive-definite real case.

Theorem 2.6. *The following results hold*

1. *The following are exact sequences*

$$1 \longrightarrow \{-1,1\} \longrightarrow \mathrm{Pin}(d,\mathbb{C}) \xrightarrow{\widetilde{\mathrm{Ad}}} O(d,\mathbb{C}) \longrightarrow 1\,, \tag{2.16}$$
$$1 \longrightarrow \{-1,1\} \longrightarrow \mathrm{Spin}(d,\mathbb{C}) \xrightarrow{\widetilde{\mathrm{Ad}}} SO(d,\mathbb{C}) \longrightarrow 1\,, \tag{2.17}$$

   and $\widetilde{\mathrm{Ad}}$ *is a (double) covering map in each case.*

2. *The complex special orthogonal group $SO(d,\mathbb{C})$ is connected. Hence, it is the identity connected component of the complex orthogonal group $O(d,\mathbb{C})$.*

3. *The complex spin group* $\mathrm{Spin}(d,\mathbb{C})$ *is connected except in the case $d = 1$ where instead* $\mathrm{Spin}(1,\mathbb{C}) = \{-1,1\}$. *Hence, for $d \geq 2$ it is the identity connected component of the pin group and the double coverings are non-trivial.*

4. *We have $\pi_1(SO(1,\mathbb{C})) = 1$, $\pi_1(SO(2,\mathbb{C})) \cong \mathbb{Z}$, $\pi_1(SO(n,\mathbb{C})) \cong \mathbb{Z}_2$ for $n \geq 3$. Consequently, due to the non-trivial double covering* $\pi_1(\mathrm{Spin}(2,\mathbb{C})) \cong \mathbb{Z}$, $\pi_1(\mathrm{Spin}(n,\mathbb{C})) \cong 1$ *for $n \geq 3$. Hence, the covering is typically universal.*

Proof. We check 1.

Nothing notable changes from the real case, we instead argue that $\ker \widetilde{\mathrm{Ad}}$ restricted to complex pin or spin groups is formed by $x = \lambda \cdot \mathrm{id}$ such that $N(x) = \lambda^2 = 1$. We skip proving local homeomorphism again.

We check 2.

Here we need to be a little careful. Reflections σ_v can only be generated by vectors v with non-zero norm $\delta(v) = \delta(v, v)$, but zero norms can occur (and occur quite frequently), e.g. in the standard basis e_a with $\delta = \mathrm{diag}(1, \ldots, 1)$, the vector $v = e_1 + ie_2$ has zero norm. First note that almost any $v \in V$ with norm $\delta(v) \neq 0$ can be connected to a unit vector with norm $+1$ by the path (principal square root)

$$\gamma(t) = \left(1 + t\left(\frac{1}{\sqrt{\delta(v)}} - 1\right)\right) v\,. \tag{2.18}$$

Indeed, trouble only arises when $\delta(v) > 0$ is real and positive since $\delta(\gamma(t)) = 0$ iff $\delta(v) = (1 - 1/t)^{-2}$. In such a case rescaling by $\gamma(t) = tv$ for $t \in [1, \delta(v)^{-\frac{1}{2}}]$ or $t \in [\delta(v)^{-\frac{1}{2}}, 1]$ will give the desired unit vector. Consider $v, w \in V$ such that $\delta(v) \neq 0 \neq \delta(w)$, then decompose $w = w_1 + w_2$ where v, w_1 are parallel and $v \perp w_2$ are orthogonal. Using the paths above, make v, w_2 unit vectors. Either $w_1 = -v$ or it doesn't, if it does, use a path that scales it and then add it to v. Now use another path to rescale $v + w_1$ back to v. Next take the path

$$\gamma(t) = tv + (1-t)w_2 \tag{2.19}$$

which clearly has $\delta(\gamma(t)) = t^2\delta(v) + (1-t)^2\delta(w_2) = t^2 + (1-t)^2 > 0$. Hence there exists a path between any two reflections. Consequently $SO(d, \mathbb{C})$ is path connected and hence connected.

We check 3.

If $d = 1$, then the Clifford algebra has no non-trivial even elements, hence $\mathrm{Spin}(1, \mathbb{C}) = \{-1, 1\}$.

If $d > 1$, then, like the real case $\mathrm{Spin}(d, \mathbb{C})$ is connected iff $\{-1, 1\}$ live in the same connected component. The same trick as in the real case works.

We check 4. The 1-dimensional case is trivial since $SO(1, \mathbb{C}) = 1$ is just a point. The $d > 2$ case follows because (I think) $SO(d) \subseteq SO(d, \mathbb{C})$ is a maximal compact subgroup. There is some assurance on Wikipedia. ☐

2.4 Classification of complex Clifford algebras

It is quite instructive to use our classification of real Clifford algebras to do this. That is, to complexify and use Lemma 1.23. Doing this we find a new lemma on periodicity

Lemma 2.7. *The following period holds* $\mathbb{C}\mathrm{l}(d{+}2) \cong \mathbb{C}\mathrm{l}(d) \otimes_{\mathbb{C}} \mathbb{C}\mathrm{l}(2)$. *Moreover,* $\mathbb{C}\mathrm{l}(d)^0 \cong \mathbb{C}\mathrm{l}(d{-}1)$

Proof. Complexify the real results of Lemma 1.22. In fact, all 3 of the real periods complexify into this same complex period. ☐

By explicit computation or complexification of the real results, it is not hard to see that

Proposition 2.8. *The following low-dimensional isomorphisms hold*

1. $\mathbb{C}\mathrm{l}(1) \cong \mathbb{C} \oplus \mathbb{C}\,,$
2. $\mathbb{C}\mathrm{l}(2) \cong \mathrm{Mat}_2(\mathbb{C})\,.$

Thus, noting that $\mathrm{Mat}_N(\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C} = \mathrm{Mat}_N(\mathbb{C})$, then we immediately get the classification.

Theorem 2.9 (Cartan/Bott)**.** *The complex Clifford algebras are classifed with 2-periodic behaviour in d.*

$d \pmod 2$	$\mathbb{C}(d)$	N
0	$\mathrm{Mat}_N(\mathbb{C})$	$2^{d/2}$
1	$\mathrm{Mat}_N(\mathbb{C}) \oplus \mathrm{Mat}_N(\mathbb{C})$	$2^{(d-1)/2}$

2.5 Classification of representations of complex Clifford algebras

Our ring theory results still apply here, so we end up with essentially the same theorem.

Theorem 2.10. *The complex Clifford algebras and their even subalgebras are classified, are semisimple and have exactly one or two simple modules (equivalently, complex irreps). The table below classifies them up to isomorphism.*

$d \pmod 2$	$\mathbb{C}(d)$	N	Modules
0	$\mathrm{Mat}_N(\mathbb{C})$	$2^{d/2}$	$\mathbb{C}^N$
1	$\mathrm{Mat}_N(\mathbb{C}) \oplus \mathrm{Mat}_N(\mathbb{C})$	$2^{(d-1)/2}$	$\mathbb{C}_1^N, \mathbb{C}_2^N$
$d \pmod 2$	$\mathbb{C}(d)^0$	N	Modules
1	$\mathrm{Mat}_N(\mathbb{C})$	$2^{(d-1)/2}$	$\mathbb{C}^N$
0	$\mathrm{Mat}_N(\mathbb{C}) \oplus \mathrm{Mat}_N(\mathbb{C})$	$2^{(d-2)/2}$	$\mathbb{C}_1^N, \mathbb{C}_2^N$

Definition 2.11. Like the real case we call these irreps as follows.

1. The irreps of $\mathbb{C}l(d)$ are called *complex pinor irreps.*

2. The irreps of $\mathbb{C}l(d)^0$ are called *complex spinor irreps.*

Notice that the real and quaternionic structures we observed for representations of the real algebras has disappeared! The representation theory is then much simpler. However, as we will soon see, real and quaternionic structures are the essence of the physics notion of *charge conjugation* and of Majorana conditions. This makes some sense from the physics viewpoint because Majorana conditions are loosely spoken about as 'reality conditions on spinors'.

Comment about how $\mathbb{C}(d) \cong \mathrm{Cl}(p,q)_{\mathbb{C}} \cong \mathrm{Cl}(p',q')_{\mathbb{C}}$ and Wick rotations at some point...

3 Analysis of Clifford algebra representations

To analyse the real representation theory of real Clifford algebras it will be convenient to complexify, as we can get some very concrete results of what a complex rep of real or quaternionic type means in terms of complex bilinear forms. Before we present this we restate two key results from earlier about complexification.

Proposition 3.1. *The following hold:*

1. *Let A be a $\mathbb{R}$-algebra, then any complex rep V gives a complex rep of its complexification $A_{\mathbb{C}}$. Moreover, by restriction we can take a complex rep of $A_{\mathbb{C}}$ to one for A. These operations are inverse.*

2. *Let A be a $\mathbb{R}$-algebra (group G), then any real rep V of A (G) can be realised as a 'real structure' on a complex representation $V_{\mathbb{C}} = V \otimes \mathbb{C}$.*

We now move onto the main theorem about complex bilinear forms and reps of real or quaternionic type. First we need to define finite groups $G_{p,q}$ with 2^{p+q+1} elements. They provide a way to almost realise real or complex Clifford algebras as group algebras.

Definition 3.2 (Gamma group)**.** Let the *gamma group* $G_{p,q}$ be defined by the generators $\{\mathrm{id}, -1, \Gamma_a\}$ obeying the relations[11]

- $(-1)^2 = \mathrm{id}$ and $(-1)\Gamma_a = \Gamma_a(-1)$,
- $(\Gamma_a)^2 = -1$ if $a = 0, \ldots, p-1$,
- $(\Gamma_a)^2 = \mathrm{id}$ if $a = p, \ldots, p+q-1$,
- $\Gamma_a\Gamma_b = (-1)\Gamma_b\Gamma_a$ if $a \neq b$.

This presentation ensures the group has 2^{p+q+1} elements all uniquely writable as

$$g = (-1)^n \Gamma_{a_1} \ldots \Gamma_{a_r} \,, \tag{3.1}$$

with $a_1 < \cdots < a_r$ and $n = 0, 1$.

3.1 Main theorem

Here is the main theorem.

Theorem 3.3. *We divide into two parts.*

1. *The Clifford algebras are (almost) group algebras in the sense that there is an ideal* $I = \langle e_{\mathrm{id}} + e_{-1} \rangle$ *such that*
   (a) $\mathrm{Cl}(p,q) \cong \mathbb{R}[G_{p,q}]/I$.
   (b) $\mathbb{C}\mathrm{l}(d) \cong \mathbb{C}[G_{d,0}]/I \cong \mathbb{C}[G_{p,q}]/I$.
2. *Let G be a compact group and V a complex rep. Then V is of real (quaternionic) type iff V has a G-invariant non-degenerate symmetric (antisymmetric) complex bilinear form B.*

   Moreover, if a real (quaternionic) structure J exists on V, then there is a choice of B and an invariant inner product $\langle -, - \rangle$ such that

$$B(v, w) = \langle J(v), w \rangle \,. \tag{3.2}$$

Proof. We check the first part about group algebras.

The group algebra $k[G_{p,q}]$ is almost a Clifford algebra, but it needs to be quotiented by the ideal $I = \langle e_{\mathrm{id}} + e_{-1} \rangle$ so that $e_{-1} \sim -1$ are treated the same. Hence, $\mathrm{Cl}(p,q) \cong \mathbb{R}[G_{p,q}]/I$ and $\mathbb{C}\mathrm{l}(d) \cong \mathbb{C}[G_{d,0}]/I \cong \mathbb{C}[G_{p,q}]/I$.

We check the second part about complex reps of compact groups.[12]

($\Longrightarrow$). Take V to be of real type with real structure J. Take some invariant complex inner product $\langle -, - \rangle$ on V. Define a symmetric complex bilinear form B by

$$B(v, w) := \frac{1}{2} \left(\langle J(v), w \rangle + \langle J(w), v \rangle \right) \,. \tag{3.3}$$

It is clear that B is invariant because $Jg = gJ$ is an intertwiner and $\langle -, - \rangle$ is invariant. Moreover, it is easy to see that B is non-degenerate by using the isomorphism $J : V \to \bar{V}$ again. Indeed, the conjugate-bilinear form

$$B(J(v), w) = \frac{1}{2} \left(\langle v, w \rangle + \langle J(w), J(v) \rangle \right) \,, \tag{3.4}$$

[11] Following the wikipedia page on higher-dimensional gamma matrices (almost).

[12] A major part of the proof is using the Haar measure to average real, complex and quaternionic inner products into invariant inner products.

is clearly also an inner product and hence non-degenerate, so B must be non-degenerate by invertibility of J.

The case of quaternionic type can be argued in much the same way using $J^2 = -\,\mathrm{id}$ and defining instead the antisymmetric bilinear form

$$B(v,w) := \frac{1}{2}\left(\langle J(w), v\rangle - \langle J(v), w\rangle\right) . \tag{3.5}$$

Again, $B(J(v), w)$ takes the same form and is an inner product so that B is again non-degenerate.

($\Longleftarrow$). Take V to have a non-degenerate symmetric complex bilinear form B and denote an invariant inner product by $\langle -, -\rangle$. Leveraging non-degeneracy of B and $\langle -, -\rangle$, define an invertible conjugate linear map $J : V \to \bar{V}$ by[13]

$$\langle v, w\rangle = B(J(v), w) . \tag{3.6}$$

We immediately see that $Jg = gJ$ is an intertwiner because

$$\langle J(v), w\rangle = B(v,w) = B(gv, gw) = \langle gg^{-1}J(gv), gw\rangle = \langle g^{-1}Jg(v), w\rangle , \tag{3.7}$$

and $\langle -, -\rangle$ is of course non-degenerate. Consider now the invertible linear map $J^2 : V \to V$. We want to show it is diagonalisable and that its eigenvalues are real and positive. First note that J is conjugate self-adjoint, i.e.

$$\langle J(v), w\rangle = B(v,w) = B(w,v) = \langle J(w), v\rangle = \overline{\langle v, J(w)\rangle} \tag{3.8}$$

Thus, J^2 is self-adjoint and hence, in any basis of V, is represented by a Hermitian matrix. Such a matrix is diagonalisable with real eigenvalues. Over this diagonalisation, any eigenvector v of J^2 obeys

$$\langle J(v), J(v)\rangle = \overline{\langle v, J^2(v)\rangle} = \bar{\lambda}\langle v, v\rangle > 0 . \tag{3.9}$$

Thus, all eigenvalues λ are real and positive. Note that each eigenspace of J^2 is a subrep because $J^2 g = gJ^2$ is an intertwiner. Hence we can define a new intertwiner $J' := \frac{1}{\sqrt{\lambda}}J$ on each eigenspace so that $J'^2 = \mathrm{id}$ makes V into a complex rep of real type.

The quaternionic case is much the same just replace B by an antisymmetric bilinear form and use the minus sign it will give (only difference ends up being that the eigenvalues are negative). Hence, it is easy to build an intertwiner J' such that $J'^2 = -\,\mathrm{id}$.

Finally, to show that, given a structure J, the existence of a choice of B and $\langle -, -\rangle_0$ such that $B(v,w) = \langle J(v), w\rangle_0$ is easy, just use the ($\Longrightarrow$) proof with

$$\langle v, w\rangle_0 := \frac{1}{2}\left(\langle v, w\rangle + \langle J(v), J(w)\rangle\right) . \tag{3.10}$$

□

A simple accompanying result is.

Proposition 3.4. *Let A_1, A_2 be $\mathbb{R}$-algebras (G_1, G_2 groups) and let V_1, V_2 be complex reps of A_1, A_2 (G_1, G_2) respectively, then*

1. *If V_1, V_2 are of quaternionic type, then $V_1 \otimes V_2$ is of real type.*
2. *If V_1, V_2 are of real type, then $V_1 \otimes V_2$ is of real type.*
3. *If V_1 is of quaternionic type and V_2 is of real type, then $V_1 \otimes V_2$ is of quaternionic type.*

[13]This kind of trick also leverages finite-dimensionality of V. Indeed, this definition is equivalent to J being determined in terms of the invertible matrices of $\langle -, -\rangle$, B (after one chooses a real basis).

Proof. A real (quaternionic) structure on a complex vector space V is just a linear map $J : V \to \bar{V}$.[14] Hence, given V_1, V_2 with J_1, J_2 structures, $V_1 \otimes V_2$ admits a natural structure $J = J_1 \otimes J_2$ given by the tensor product functor. Clearly, J is conjugate-linear

$$J(\lambda v_1 \otimes v_2) = J_1(\lambda v_1) \otimes J_2(v_2) = \bar{\lambda} J_1(v_1) \otimes J_2(v_2)\,, \tag{3.11}$$

Moreover, $J^2 = J_1^2 \otimes J_2^2$ determines whether it is a real or quaternionic structure in the sense that the proposition states. Finally, $J = J_1 \otimes J_2$ obviously preserves the rep because J_1, J_2 do. $\square$

3.2 Charge conjugation

We can apply the theorem to the real irreps of $\mathrm{Cl}(p,q)$, indeed, all we need to do is:

1. If real irrep ($V \cong \mathbb{R}^N$) with no extra structure, then complexify to the complex irrep $V_\mathbb{C}$ of real type and apply theorem to get B, J and $\langle -, - \rangle$.
2. If complex irrep ($V \cong \mathbb{C}^N$) with no extra structure, then there isn't any B or J, but we still have $\langle -, - \rangle$
3. If quaternionic irrep ($V \cong \mathbb{H}^N$), then theorem already applies.

Assuming we have complexified if needed, then a real irrep of $\mathrm{Cl}(p,q)$ has:

1. An invariant inner product $\langle -, - \rangle$.
2. If of real (quaternionic) type, an invariant complex bilinear form B which is symmetric (antisymmetric) such that $B(v,w) = \langle J(v), w \rangle$.

Where, by invariance, we mean that it is a rep of $G_{p,q}$ with $-1 \mapsto -\mathrm{id}$ so that

$$\langle \Gamma_a v, \Gamma_a w \rangle = \langle v, w \rangle\,, \tag{3.12}$$

$$B(\Gamma_a v, \Gamma_a w) = B(v, w)\,. \tag{3.13}$$

This is quite useful, indeed, if we pick a basis of the rep and let $\langle -, - \rangle$ be represented by the matrix A and let B be represented by the matrix C, then the above is equivalent to

$$\Gamma_a^\dagger = A \Gamma_a^{-1} A^{-1}\,, \tag{3.14}$$

$$\Gamma_a^T = C \Gamma_a^{-1} C^{-1}\,, \tag{3.15}$$

where $\Gamma_a^{-1} = \pm \Gamma_a$ depending on $\Gamma_a^2 = \pm 1$.

In physics,

4 Wick rotations

A Clifford algebra (almost) as a group algebra of a finite group

We detail how Clifford algebras are (almost) group algebras for a finite group G. In particular, this means we can translate results of the rep theory of finite groups over to that of Clifford algebras.

A key point about finite groups G is that their reps over a field k are in correspondence (bijection) with reps of the k-group algebra $k[G] = \mathrm{span}(e_g)_{g \in G}$.

A further point is if $k_1 \subset k_2$ is a subfield of k_2 then k_2-representations of G are in correspondence with k_2-representations of $k_1[G]$. So we may take $k_1 = \mathbb{R}$, $k_2 = \mathbb{C}$ if we want.

[14] I said 'conjugate-linear' many times in definitions so far to emphasize it, but in reality $J : V \to \bar{V}$ is linear. The conjugation is dealt with by the structure of $\bar{V}$. Remember $V = \bar{V}$ as sets, it is only in the category of complex vector spaces that they are distinguished by $\lambda \cdot v = \bar{\lambda} v$ for $v \in \bar{V}$.

Gamma group. One can *almost* redefine $\mathrm{Cl}(p,q)$ as the real group algebra generated by the finite group G_Γ defined by the generators $\{\mathrm{id}, -1, \Gamma_a\}$[15] obeying the relations

- $(-1)^2 = \mathrm{id}$ and $(-1)\Gamma_a = \Gamma_a(-1)$,
- $(\Gamma_a)^2 = -1$ if $a = 0, \ldots, p-1$,
- $(\Gamma_a)^2 = \mathrm{id}$ if $a = p, \ldots, p+q-1$,
- $\Gamma_a\Gamma_b = (-1)\Gamma_b\Gamma_a$ if $a \neq b$.

Due to these relations we find all elements $g \in G_\Gamma$ can be written uniquely as

$$g = (-1)^n \Gamma_{a_1} \ldots \Gamma_{a_m} \tag{A.1}$$

where $n = 0, 1$ and $a_1 < \cdots < a_m$ and $0 \leq m \leq p+q$. Thus this is indeed a finite group and has $|G_\Gamma| = 2^{p+q+1}$ elements since

$$2^N = \sum_{i=0}^{N} \binom{N}{i} . \tag{A.2}$$

Group algebra and quotient. Indeed, the group algebra $k[G]$ over a field k of some group G has basis $\{e_g\}$. Thus we have the issue that e_{-1} is considered a distinct element. This is easy to fix, we just quotient by the ideal $I = \langle e_{-1} + e_{\mathrm{id}} \rangle$, i.e. set $e_{-1} = -e_{\mathrm{id}}$, and get

$$\mathbb{R}[G_\Gamma]/I = \mathrm{Cl}(p,q) . \tag{A.3}$$

Representations. We only consider reps $\rho : G_\Gamma \to GL(V)$ such that $\rho(\alpha) = -\rho(\mathrm{id})$. Such reps obviously extend to reps of $\mathbb{R}[G_\Gamma]/\langle e_\alpha = -e_{\mathrm{id}} \rangle = \mathrm{Cl}(p,q)$ and vice-versa.

The main point to all of this is that we can leverage some useful results of the rep theory of finite groups.

Lemma A.1. *Given a complex rep V of a finite group G, there exists a G-invariant inner product on V.*

Proof. Take any inner product $\langle -, - \rangle$ on V, then a G-invariant inner product $\langle -, - \rangle_G$ is given by

$$\langle v, w \rangle_G := \sum_{g \in G} \langle gv, gw \rangle \tag{A.4}$$

where $gv := \rho(g)v$ is shorthand. □

As a consequence we get Maschke's theorem so that representations V always decompose into the direct sum of irreps.

Comment: This 'averaging' trick will not always work. For example, a symmetric bilinear form can be averaged to a G-invariant symmetric bilinear form and so can an antisymmetric bilinear form. However, non-degeneracy is not always preserved. Indeed, the reason inner products work is because *positive-definite* can be preserved.

Other useful things about finite groups that are worth mentioning (may or may not make use of these results):

1. Frobenius-Schur indicator
2. Classification of complex irreps
3. Character theory in general

[15] Following the wikipedia page on higher-dimensional gamma matrices (almost).

B Extras

A less intuitive more convoluted way to prove part of Theorem 3.3 Finally, we generalise to V being any rep. The crux of the generalisation comes from the ($\Longleftarrow$) direction. Here we give V a G-invariant complex inner product $\langle -, - \rangle$ by averaging and note that, through its use, V decomposes into orthogonal irreps

$$V \cong \bigoplus_i n_i V_i \,, \tag{B.1}$$

where $n_i V_i = V_i \oplus \cdots \oplus V_i$ is a sum of isotypic components, i.e. they are all isomorphic as irreps. In particular, we can assume that the inner product is the same on each V_i in the isotypic decomposition, which will help soon. The point is then that $J^2 : V \to V$ is still an isomorphism of reps because the same definition of J as

$$\langle v, w \rangle = B(J(v), w) \tag{B.2}$$

ensures it is invertible, conjugate-linear and an intertwiner. We now work with individual isotypic components because Schur's lemma ensures that J^2 restricted to an isotypic component $n_i V_i$ will map into itself, i.e. $J^2 : n_i V_i \to n_i V_i$. To simplify notation, pick any isotypic component and denote it nV. If the linear map J^2 is restricted to the j^{th}-subspace V of nV, then Schur's lemma also ensures that it is of the form

$$J^2(v_j) = J^2 \underbrace{(0, \ldots, v, \ldots, 0)}_{v \text{ in } j^{\text{th}}\text{-position}} = (\lambda_1^j v, \ldots, \lambda_n^j v) \,, \tag{B.3}$$

for some constants $\lambda_k^j \in \mathbb{C}$. Indeed, when considering the restriction of J^2 on some isotypic component nV, we find that

$$J^2 = \begin{pmatrix} \lambda_1^1 & \ldots & \lambda_1^n \\ \vdots & \ddots & \vdots \\ \lambda_{n_i}^1 & \ldots & \lambda_n^n \end{pmatrix} , \qquad v_j = (0, \ldots, v, \ldots, 0) = \begin{pmatrix} 0 \\ \vdots \\ v \\ \vdots \\ 0 \end{pmatrix} \tag{B.4}$$

Working in the case where B is symmetric we can use orthogonality to see that the matrix for J^2 is symmetric because

$$\bar{\lambda}_k^j \langle v, v \rangle = \langle J^2(v_j), v_k \rangle = \langle v_j, J^2(v_k) \rangle = \lambda_j^k \langle v, v \rangle \,. \tag{B.5}$$

Thus, J^2 is a Hermitian matrix. Consequently, there exists a basis (in fact many bases) of nV which diagonalises J^2 on nV and its eigenspaces will be isomorphic to V. That is, we have picked a new decomposition of nV as a direct sum of identical irreps V. In this new decomposition we can calculate that diagonal values are positive

$$\langle J(v_j), J(v_j) \rangle = \overline{\langle v_j, J^2(v_j) \rangle} = \bar{\lambda}_j^j \langle v_j, v_j \rangle > 0 \,. \tag{B.6}$$

As $J^2 g = g J^2$, then the eigenspaces $V_\lambda \cong V$ of J^2 are preserved and we can define $J' : nV \to nV$ by $J' = \lambda^{-1/2} J$ on each. Clearly, $J'^2 = \mathrm{id}$ on nV and retains conjugate-linearity and G-invariance. The same argument goes through for each isotypic component, so $J'^2 = \mathrm{id}$ on the whole rep.

Very similar reasoning applies to the case where B is antisymmetric, the only difference is that

$$\langle J(v_j), J(v_j) \rangle = -\overline{\langle v_j, J^2(v_j) \rangle} = -\bar{\lambda}_j^j \langle v_j, v_j \rangle > 0 \,, \tag{B.7}$$

makes the eigenvalues of J all negative. Hence we get J' defined in the same way defined as $J'^2 = -\mathrm{id}$.

In the above convoluted proof we made use of the following form of Schur's lemma.

Lemma B.1 (Schur's lemma for algebraically closed fields)**.** *If V is a complex irrep of a group G, then there is a canonical $\mathbb{C}$-algebra isomorphism $\mathrm{End}_G(nV) \cong \mathrm{Mat}_n(\mathbb{C})$. Moreover, $\mathrm{Hom}_G(nV, mV)$ will take a similar form, but only as a $\mathbb{C}$-vector space of rectangular matrices. Moreover,*

1. *This works also for algebras A.*
2. *The same result holds for reps of G over any algebraically closed field.*

Proof. Schur's lemma for algebraically closed fields ensures us that $\mathrm{End}_G(V) = \mathbb{C}$.

Consider an intertwiner $\varphi : V \oplus V \to V \oplus V$. As a linear map $\phi \in \mathrm{Hom}(V \oplus V, V \oplus V)$ decomposes into 4 pieces since

$$\mathrm{Hom}(V \oplus V, V \oplus V) \cong \mathrm{Hom}(V, V \oplus V) \oplus \mathrm{Hom}(V, V \oplus V) \tag{B.8}$$
$$\cong \mathrm{Hom}(V, V) \oplus \mathrm{Hom}(V, V) \oplus \mathrm{Hom}(V, V) \oplus \mathrm{Hom}(V, V)\,. \tag{B.9}$$

The first decomposition is obtained by restriction and the second is obtained by restriction followed by projection.[16] Restriction clearly does not affect the intertwiner property $g\phi = \phi g$ and neither does projection because the group action $g(v, w) = (gv, gw)$ is the same before or after a projection. Hence, the above decomposition holds in terms of intertwiners in the sense

$$\mathrm{Hom}_G(V \oplus V, V \oplus V) \cong \mathrm{Hom}_G(V, V) \oplus \mathrm{Hom}_G(V, V) \oplus \mathrm{Hom}_G(V, V) \oplus \mathrm{Hom}_G(V, V) \tag{B.10}$$
$$= \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}\,. \tag{B.11}$$

At this point it is important to emphasize that all the isomorphisms used to decompose $\mathrm{Hom}(-,-)$ were canonical (we just used restriction and projection maps). Hence, we do indeed have a canonical isomorphism

$$\mathrm{End}_G(V \oplus V) \cong \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}\,, \tag{B.12}$$

which equivalently states that

$$\phi(v, w) = (av + bw, cv + dw) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}\,, \tag{B.13}$$

for some $a, b, c, d \in \mathbb{C}$. It is easy to see that

$$\phi'\phi(v, w) = (a'(av + bw) + b'(cv + dw), c'(av + bw) + d'(cv + dw)) \tag{B.14}$$
$$= ((a'a + b'c)v + (a'b + b'd)w, (c'a + d'c)v + (c'b + d'd)w) \tag{B.15}$$
$$= \begin{pmatrix} a'a + b'c & a'b + b'd \\ c'a + d'c & c'b + d'd \end{pmatrix} \tag{B.16}$$
$$= \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}\,. \tag{B.17}$$

Hence, we have a natural isomorphism $\mathrm{End}_G(V \oplus V) \cong \mathrm{Mat}_2(\mathbb{C})$. This argument extends quite generally to $\mathrm{Hom}_G(nV, mV)$.

This argument does not change for algebras, nor does it change for algebraically closed fields other than $\mathbb{C}$. □

[16] This can be equivalently spoken about using universal properties of products and coproducts of which $\oplus$ has both.

Another convoluted but interesting proof of part of Theorem 3.3 We check 2.

($\implies$). Take V to be of quaternionic type with quaternionic structure J. Then $V \cong \mathbb{H}^N$ can be treated as a left $\mathbb{H}$-module with $J(v) \equiv jv$ and a right $\mathbb{H}$-module with $J(v) \equiv v(-j)$. In particular, we can pick a complex basis of V given by $\{e_1, \ldots, e_N, J(e_1), \ldots, J(e_N)\}$ which gives a quaternionic basis $\{e_1, \ldots, e_N\}$ of V. There is then an invariant quaternionic inner product $\langle -, - \rangle$ on V obtained by averaging the standard one $\langle -, - \rangle_s$ on right modules $\mathbb{H}^N$[17]

$$\langle v, w \rangle_s = \langle v^i e_i, w^j e_j \rangle_s = \sum_i \bar{v}^i w^i \,. \tag{B.18}$$

As one can check $\langle v, w \rangle_s = \overline{\langle w, v \rangle}_s$ and $\langle v, wq \rangle = \langle v, w \rangle q$ and $\langle -, - \rangle$ inherits this.

This quaternionic inner product can be broken into two complex-valued parts H and B

$$\langle v, w \rangle = H(v, w) + B(v, w) j \,. \tag{B.19}$$

As $\langle v, w \rangle = \overline{\langle w, v \rangle}$, then,

$$\begin{aligned} H(v, w) + B(v, w) j &= \overline{H(w, v)} + \overline{B(w, v) j} && \text{(B.20)} \\ &= \overline{H(w, v)} + \bar{j} \overline{B(w, v)} && \text{(B.21)} \\ &= \overline{H(w, v)} - j \overline{B(w, v)} && \text{(B.22)} \\ &= \overline{H(w, v)} - B(w, v) j \,. && \text{(B.23)} \end{aligned}$$

Thus, we see that $H(v, w) = \overline{H(w, v)}$ and $B(v, w) = -B(w, v)$. Moreover, $\langle v, wq \rangle = \langle v, w \rangle q$ ensures that both H, B inherit this property too. Thinking of V as just a complex vector space shows H to be a hermitian form and B to be an antisymmetric bilinear form. It is clear that H is actually a complex inner product since $\langle v, v \rangle = H(v, v) > 0$ for all non-zero $v \in V$.

Moreover, B is non-degenerate because, if it was not, then there exists v_0 such that $B(v_0, v) = 0$ for all $v \in V$, which forces $\langle v_0, v \rangle = H(v_0, v)$ to be complex-valued. However, right $\mathbb{H}$-linearity tells us that for $vj \in V$ we have $\langle v_0, vj \rangle = \langle v_0, v \rangle j$. The only conclusion is then that $\langle v_0, v \rangle = 0$ for all $v \in V$, but this contradicts that $\langle -, - \rangle$ is a quaternionic inner product.

[17] I distinguish right modules here because $\langle v, wq \rangle = \langle v, w \rangle q$, but there is no nice version of this for left actions of q. Instead, one would need to pick the other inner product summing over $v^i \bar{w}^i$, at which point left actions are nice and right actions are not.