Clifford algebra and Wick rotation notes

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Contents

These notes make use of

- José Figueroa-o'Farill's online notes (Majorana.pdf) on Clifford algebras and their representations
- Fulton and Harris – Representation Theory: A First Course
- Jean Gallier's online notes (clifford.pdf) on Clifford algebras
- The wikipedia page on higher-dimensional gamma matrices
- The lecture notes on Clifford algebras by Lundholm and Svensson.
- The wikipedia page on the spin group
- Varadarajan - Supersymmetry for mathematicians
- Many posts on math stackexchange and math overflow

and other resources. There are some errors or bad exposition in some of the above resources, and (to the best of my ability) there are rewrites/corrections here. We focus mainly on the case of real/complex Clifford algebras (as we seek to characterise Wick rotation eventually), but a decent amount of techniques will readily extend to fields other than $\mathbb{R}, \mathbb{C}$.

1 Structure of real Clifford algebras $\mathrm{Cl}(p, q)$

1.1 Definitions and conventions

Let $V = \mathbb{R}^{p,q}$ be endowed with the metric $\eta_{ab} = \mathrm{diag}(1, \dots, -1)$. We will take the Clifford algebra $\mathrm{Cl}(p, q)$ to be generated by products of Γ_a obeying

$$\Gamma_a \Gamma_b + \Gamma_b \Gamma_a = -2\eta_{ab}\,\mathrm{id}\ . \tag{1.1}$$

If one wants to identify this with the construction by quotienting the tensor algebra of $\mathbb{R}^{p,q}$, then Γ_a correspond to the usual basis vectors e_a of $\mathbb{R}^{p,q}$ such that $\eta(e_a, e_b) = \eta_{ab}$ and one writes instead

$$e_a e_b + e_b e_a = -2\eta_{ab}\,\mathrm{id}\ . \tag{1.2}$$

This formulation also reveals the natural inclusion of the underlying vector space $\mathbb{R}^{p,q} \to \mathrm{Cl}(p, q)$. A basis of $\mathrm{Cl}(p, q)$ is then given by the 2^{p+q} vectors $\{\mathrm{id}, e_{a_1} \dots e_{a_n}\}$ where $a_1 < \dots < a_n$ and $n = 1, \dots, p + q$, very analogous to the exterior algebra $\Lambda(\mathbb{R}^{p,q})$.

We briefly mention that the complex Clifford algebra $\mathbb{C}\mathrm{l}(d)$ of $\mathbb{C}^d$ endowed with the Euclidean bilinear form $\delta = \mathrm{diag}(1, \dots, 1)$ is isomorphic to the complexification $\mathrm{Cl}(p, q)$ of any real Clifford algebra with $d = p + q$.

1.2 Natural maps

We now look at some natural linear maps $\Phi : \mathrm{Cl}(p, q) \to \mathrm{Cl}(p, q)$ which preserve the multiplication defined by (1.2). The 'natural' part means that it is independent of the basis we originally choose for $V \cong \mathbb{R}^{p,q}$. Hence we will work explicitly in a basis-independent setting with $\mathrm{Cl}(V, \eta)$ defined by its construction as a quotient of the tensor algebra. Here we just have a natural inclusion map $V \to \mathrm{Cl}(V, \eta)$ and the relation

$$vw + wv = -2\eta(v, w) \tag{1.3}$$

for some symmetric non-degenerate bilinear form η with signature (p, q). Such maps Φ then obey

$$\Phi(vw + wv) = vw + wv = -2\eta(v, w)\,, \tag{1.4}$$

where we additionally require that Φ restricts to a linear map $V \to V$. Sometimes these maps are entirely determined by their restriction $V \to V$ and the rule $\Phi(vw) = \Phi(v)\Phi(w)$. In this case the condition (1.4) can be rewritten as $\Phi \in O(V, \eta)$.

1.2.1 Signature-independent maps

Two important examples are the *main involution* α and the *reversal* or *main anti-automorphism* τ defined by

$$\alpha(v) := -v\,, \tag{1.5}$$

$$\tau(v_1 \dots v_n) := v_n \dots v_1\,, \tag{1.6}$$

where $v, v_1, \dots, v_n \in V$ are arbitrary vectors. Indeed, we can see that

$$\alpha(vw + wv) = \alpha(v)\alpha(w) + \alpha(w)\alpha(v) = (-1)^2(vw + wv)\,, \tag{1.7}$$

$$\tau(vw + wv) = \tau(vw) + \tau(wv) \qquad = wv + vw\,, \tag{1.8}$$

both preserve Clifford multiplication. Note that $\tau(xy) = \tau(y)\tau(x)$ in general; this is why it is called an *anti*-automorphism.

Both maps α and τ are independent of the signature (p, q). In fact, $\alpha : V \to V$ which sends $v \mapsto -v$ is just an orthogonal linear map on V which we have lifted to the Clifford algebra. Physically, $\alpha = PT \in O(V, \eta)$ which flips both space and time directions.

1.2.2 A signature-dependent map

An involution β which depends on the signature (p, q) can be constructed starting from a map $\beta : V \to V$ which depends on the signature. Indeed, we just want to choose $\beta = P$ or $\beta = T$. The interpretation depends on a choice of what part of V is temporal and what part of V is spatial, i.e. a choice of decomposition $V = V_1 \oplus V_2$ where $V_1 \perp V_2$ are orthogonal and $\eta|_{V_1}$ is positive-definite and $\eta|_{V_2}$ is negative-definite.[1] With the choice made, define

$$\beta(v_1) := v_1\,, \tag{1.9}$$

$$\beta(v_2) := -v_2\,. \tag{1.10}$$

Sadly, such a map is not natural (basis-independent) because we had to choose a decomposition $V = V_1 \oplus V_2$, of which there are infinitely many, even in the case $\dim V = 2$, $(p, q) = (1, 1)$. However, it is independent of a choice of basis for V_1 and for V_2. Hence one could at least say it is natural for $\mathrm{Cl}(V_1 \oplus V_2, \eta)$.

We can also readily define $\gamma = \alpha\beta = \beta\alpha$ which acts as T if β acts as P. Explicitly,

$$\gamma(v_1) := -v_1\,, \tag{1.11}$$

$$\gamma(v_2) := v_2\,. \tag{1.12}$$

These maps β, γ essentially correspond to the complex transpose of a given matrix representation of the Clifford algebra. We will make this more precise later and relate it to pinor and spinor representations.

1.2.3 Clifford algebras are superalgebras

A useful point is that $\alpha(v) = -v$ clearly decomposes $\mathrm{Cl}(V, \eta)$ into two parts

$$\mathrm{Cl}(V, \eta) = \mathrm{Cl}(V, \eta)^0 \oplus \mathrm{Cl}(V, \eta)^1\,, \tag{1.13}$$

[1] We take V_1 to be temporal, so $\beta = P$.

where $\mathrm{Cl}(V,\eta)^0$ contains linear combinations of even products and $\mathrm{Cl}(V,\eta)^1$ contains linear combinations of odd products.

It is obvious that Clifford multiplication preserves this $\mathbb{Z}_2$-grading, i.e.

$$\mathrm{Cl}(V,\eta)^i \oplus \mathrm{Cl}(V,\eta)^j \subseteq \mathrm{Cl}(V,\eta)^{i+j}\,, \tag{1.14}$$

where $i+j$ is interpreted mod 2. In particular, the product of two evens is even, the product of two odds is even and the product of an even and an odd is odd.

An interesting point is that any subset of $\mathrm{Cl}(V,\eta)$ will naturally inherit these grading properties, including the group of units $\mathrm{Cl}(V,\eta)^\times$. One could call it a 'supergroup', but this is bad terminology as it typically means a super Lie group, i.e. a group in the category of supermanifolds.

1.3 Lipschitz–Clifford group and pin and spin groups

We first consider how to realise reflections in a Clifford algebraic way. Then develop the tools needed to define pin and spin groups as groups living inside $\mathrm{Cl}(V,\eta)$.

The idea is that we will first establish a group $\Gamma(V,\eta) \subseteq \mathrm{Cl}(V,\eta)$ called the *Lipschitz–Clifford group* or *Lipshitz group* or *Clifford group* which has a representation on V

$$\widetilde{\mathrm{Ad}} : \Gamma(V,\eta) \to GL(V) \tag{1.15}$$

such that its image contains all possible compositions of reflections and, by Cartan–Dieudonné theorem, the entire orthgonal group, i.e. $O(V,\eta) \subseteq \mathrm{im}\,\widetilde{\mathrm{Ad}}$ (in fact $O(V,\eta) = \mathrm{im}\,\widetilde{\mathrm{Ad}}$). We will then find a method to identify the appropriate subgroup $\mathrm{Pin}(V,\eta) \subseteq \Gamma(V,\eta)$ corresponding to the Pin group, at which point we will have the representation

$$\widetilde{\mathrm{Ad}} : \mathrm{Pin}(V,\eta) \to O(V,\eta) \tag{1.16}$$

which is also a covering map with $\mathrm{Pin}(V,\eta)$ the double (and, depending on (p,q), universal) cover of $O(V,\eta)$.

This will readily descend to the case of the even subgroup $\mathrm{Spin}(V,\eta) := \mathrm{Pin}^0(V,\eta) := \mathrm{Pin}(V,\eta) \cap \mathrm{Cl}(V,\eta)^0$ with

$$\widetilde{\mathrm{Ad}} : \mathrm{Spin}(V,\eta) \to SO(V,\eta)\,. \tag{1.17}$$

We will also define the orthochronous spin group $\mathrm{Spin}^+(V,\eta) \subseteq \mathrm{Spin}(V,\eta)$ which will always be connected, except for $(p,q) = (1,0),(0,1),(1,1)$. Equality $\mathrm{Spin}^+(V,\eta) = \mathrm{Spin}(V,\eta)$ is only obtained for definite forms η with $(p,q) = (0,d),(d,0)$.

We will also give a more explicit presentation of pin and spin as groups of the form

$$\mathrm{Pin}(V,\eta) = \{v_1 \dots v_r \mid \eta(v_i) = \pm 1\} \subseteq \mathrm{Cl}(p,q)\,, \tag{1.18}$$

$$\mathrm{Spin}(V,\eta) = \{v_1 \dots v_{2r} \mid \eta(v_i) = \pm 1\} = \mathrm{Pin}(p,q) \cap \mathrm{Cl}(p,q)^0\,, \tag{1.19}$$

where $\eta(v) := \eta(v,v)$. That is, each element is merely a product of vectors with non-zero norms.

1.3.1 Reflections

Consider some basis-free formulation (V,η). Given some $v \in V$ such that $\eta(v,v) \neq 0$, we can reflect about the hyperplane orthogonal to v by

$$\sigma_v(w) = w - 2\frac{\eta(v,w)}{\eta(v,v)}v\,. \tag{1.20}$$

Geometrically, one wants to translate w far enough in the direction v so that their 'dot product' changes signs, i.e. $\eta(\sigma_v(w),v) = -\eta(w,v)$. The following lemma gives their main properties.

Lemma 1.1. *Let $\sigma_v : V \to V$ be a reflection, then σ_v is linear, $\sigma_v \in O(V,\eta)$ and $\det(\sigma_v) = -1$.*

Proof. Reflections are obviously linear by virtue of their formula. Checking $\sigma_v \in O(V,\eta)$ just amounts to computing $\eta(\sigma_v(w), \sigma_v(u))$, so we skip this.

Checking $\det(\sigma_v) = -1$ is more interesting. Notice v determines an orthogonal decomposition $V = \mathrm{span}(v) \oplus H$. Moreover, it is clear that $\sigma_v(v) = -v$ and $\sigma_v(h) = h$. Hence, in any basis of H, σ_v has only eigenvalue 1 on each basis vector, whilst it has eigenvalue -1 on any basis of $\mathrm{span}(v)$. Thus, $\det(\sigma_v) = \prod \lambda_i = -1$. □

Finally, a key theorem is that reflections actually generate the orthogonal group!

Theorem 1.2 (Cartan–Dieudonné). *Let (V,η) be a $d = p+q$-dimensional vector space over a field k with $\mathrm{char}(k) \neq 2$ and with non-degenerate symmetric bilinear form η. Then every element of $O(V,\eta)$ is a composition of at most d reflections.*

Similarly, even products of reflections must generate the special orthogonal group.

1.3.2 The twisted adjoint representation and Clifford-Lipschitz group

If we consider reflections in context of the Clifford algebra by leveraging the natural inclusion $V \to \mathrm{Cl}(V,\eta)$, then one may write

$$\begin{aligned}
\sigma_v(w) &= w - \frac{2\eta(v,w)}{\eta(v,v)} v && (1.21)\\
&= w - \frac{-(vw+wv)}{(-v^2)} v && (1.22)\\
&= w - \frac{(vwv + wv^2)}{v^2} && (1.23)\\
&= -vwv^{-1} && (1.24)\\
&= \alpha(v) w v^{-1} && (1.25)\\
&=: \widetilde{\mathrm{Ad}}(v) w\,. && (1.26)
\end{aligned}$$

Thus, reflections $\sigma_v : V \to V$ are realised naturally by the Clifford algebra by *twisted conjugation* or *twisted adjoint representation.*

This *twisted adjoint representation* is quite an interesting concept to make more general. Let $\mathrm{Cl}(V,\eta)^\times$ denote the group of units. Let $x \in \mathrm{Cl}(V,\eta)^\times$ and $y \in \mathrm{Cl}(V,\eta)$, then we define the linear map[2]

$$\widetilde{\mathrm{Ad}}(x)y := \alpha(x) y x^{-1}\,. \tag{1.27}$$

A basic property of the twisted adjoint representation is that

$$\widetilde{\mathrm{Ad}}(x_1x_2)y = \alpha(x_1x_2)y(x_1x_2)^{-1} = \alpha(x_1)\alpha(x_2)yx_2^{-1}x_1^{-1} = \widetilde{\mathrm{Ad}}(x_1)\widetilde{\mathrm{Ad}}(x_2)y\,. \tag{1.28}$$

Consequently, $\widetilde{\mathrm{Ad}}(x)$ is invertible with inverse $\widetilde{\mathrm{Ad}}(x)^{-1} = \widetilde{\mathrm{Ad}}(x^{-1})$.

We now consider the collection of $x \in \mathrm{Cl}(V,\eta)^\times$ that act as linear maps $\widetilde{\mathrm{Ad}}(x) : V \to V$ when restricted to V.

Definition 1.3 (Lipschitz–Clifford group)**.** The *Lipschitz–Clifford group* $\Gamma(V,\eta)$ is defined as

$$\Gamma(V,\eta) := \{x \in \mathrm{Cl}(V,\eta)^\times \mid \widetilde{\mathrm{Ad}}(x)(v) \in V \quad \text{for all } v \in V\}\,. \tag{1.29}$$

Similarly, we define the *special Lipschitz–Clifford group* by $\Gamma^0(V,\eta) := \Gamma(V,\eta) \cap \mathrm{Cl}(V,\eta)^0$.

[2] This is not quite an automorphism as $\widetilde{\mathrm{Ad}}(x)(y_1y_2)$ need not always equal $\widetilde{\mathrm{Ad}}(x)(y_1)\widetilde{\mathrm{Ad}}(x)(y_2)$.

This indeed defines a group. If $x_1, x_2 \in \Gamma(V,\eta)$, then $x_1x_2 \in \Gamma(V,\eta)$ because

$$\widetilde{\mathrm{Ad}}(x_1x_2)v = \widetilde{\mathrm{Ad}}(x_1)\widetilde{\mathrm{Ad}}(x_2)v \in V\,. \tag{1.30}$$

Moreover, if $x \in \Gamma(V,\eta)$, then $x^{-1} \in \Gamma(V,\eta)$ because $\widetilde{\mathrm{Ad}}(x^{-1}) = \widetilde{\mathrm{Ad}}(x)^{-1} : V \to V$. The case of the subset $\Gamma^0(V,\eta)$ being a group follows as well since the product of evens is even.

The twisted adjoint representation is a 'surjective' representation. From the above we can see that the (special) Lipschitz–Clifford groups are afforded a representation on V by $\widetilde{\mathrm{Ad}}$. That is we have group homomorphisms

$$\widetilde{\mathrm{Ad}} : \Gamma(V,\eta) \to GL(V)\,, \tag{1.31}$$

$$\widetilde{\mathrm{Ad}} : \Gamma^0(V,\eta) \to GL(V)\,. \tag{1.32}$$

Note that arbitrary products of vectors $x = v_1 \ldots v_r$ exist in $\mathrm{Cl}(V,\eta)$. Indeed, this is fine, but such an expression is not unique in general. If $r > p+q$ we can guarantee such a product will decompose into a linear combination of products of vectors with $p+q \le r$ by picking a basis of V and using Clifford multiplication (1.2).

Such a product $x = v_1 \ldots v_r$ will also be a unit iff $\eta(v_i,v_i) \neq 0$ are all non-zero. This is because $\tau(x) = \lambda x^{-1}$ for some non-zero $\lambda \in \mathbb{R}$ iff $\eta(v_i,v_i) \neq 0$ are all non-zero. Moreover, $\widetilde{\mathrm{Ad}}(x)v \in V$ because this is just a composition of many reflections. Thus, we get the following result by leveraging Cartan-Dieudonné.

Proposition 1.4. *The twisted adjoint representation*

$$\widetilde{\mathrm{Ad}} : \Gamma(V,\eta) \to GL(V) \tag{1.33}$$

is indeed a representation and $O(V,\eta) \subseteq \mathrm{im}\,\widetilde{\mathrm{Ad}}$. *The analogous result holds for* $\Gamma^0(V,\eta)$ *and* $SO(V,\eta)$.

1.3.3 The Clifford norm and properties of the Clifford–Lipschitz group

Now we introduce a useful map κ defined by $\kappa := \alpha\tau = \tau\alpha$. A similar, but less useful map is $\kappa_\gamma := \gamma\tau = \tau\gamma$ and κ_γ, by virtue of using γ, is dependent on a choice of decomposition $V = V_1 \oplus V_2$. They have the following property when acting on products of orthonormal basis vectors

$$x\kappa(x) = \pm 1\,, \tag{1.34a}$$

$$x\kappa_\gamma(x) = 1\,, \tag{1.34b}$$

with $x = e_{a_1} \ldots e_{a_n}$.

Definition 1.5 (Clifford norm)**.** The *Clifford norm* $N : Cl(V,\eta) \to Cl(V,\eta)$ is defined by $N(x) := x\kappa(x)$.[3]

We need a technical lemma to progress.

Lemma 1.6. *The kernel of* $\widetilde{\mathrm{Ad}} : \Gamma(V,\eta) \to GL(V)$ *is* $\ker(\widetilde{\mathrm{Ad}}) = \mathbb{R}^\times \cdot \mathrm{id}$.

[3] We avoid using $N_\gamma(x) := x\kappa_\gamma(x)$ because it is unyieldly to work with and actually loses some important properties which N has.

Proof. Let $x \in \ker(\widetilde{\mathrm{Ad}})$, then $\alpha(x)v = vx$ for all $v \in V$ when treating things inside $\mathrm{Cl}(V, \eta)$.

Since v has fixed parity 1, decomposing $x = x_0 + x_1$ yields two equations

$$x_0 v = v x_0 \,, \tag{1.35}$$

$$-x_1 v = v x_1 \,. \tag{1.36}$$

Fix a basis e_a of V which diagonalises η as $\eta_{ab} = \mathrm{diag}(1, \dots, -1)$. This determines a basis $\{\mathrm{id}, e_{a_1} \dots e_{a_n}\}$ where $a_1 < \dots < a_n$ and $n = 1, \dots, p+q$ of $\mathrm{Cl}(V, \eta)$. Pick any basis vector e_a, then this guarantees that there is a unique expansion

$$x_0 = a + e_a b \tag{1.37}$$

where a is even, b is odd and both do not contain e_a. Given the expansion of x_0, the equation $x_0 v = v x_0$ then decomposes according to parity into two more equations. These are

$$av = va \,, \tag{1.38}$$

$$e_a b v = v e_a b \,. \tag{1.39}$$

Now set $v = e_a$. Since b does not contain e_a and orthogonality means $e_a e_b = -e_b e_a$ when $a \neq b$, then $b e_a = -e_a b$ since b is odd. The second equation then becomes

$$-(e_a)^2 b = (e_a)^2 b \,, \tag{1.40}$$

where $(e_a)^2 = \pm 1 \neq 0$, which sets $b = 0$. Since the choice of e_a in the expansion of x_0 was arbitrary, this means that $x_0 = \lambda\,\mathrm{id}$ for some $\lambda \in \mathbb{R}$. The same tricks applied to $-x_1 v = v x_1$ also show that $x_1 = 0$. Hence, $x = x_0 = \lambda\,\mathrm{id}$ and since x is a unit, $\lambda \in \mathbb{R}^\times$ is non-zero. □

Now we can show that the Clifford norm is well-behaved on the Lipschitz–Clifford group. It is useful to note that α, τ being (anti)automorphisms guarantees that $\alpha(x^{-1}) = \alpha(x)^{-1}$ and the same for τ.

Proposition 1.7. *The following are true*

1. *If $x \in \Gamma(V, \eta)$, then $N(x) \in \ker(\widetilde{\mathrm{Ad}}) = \mathbb{R}^\times \cdot \mathrm{id}$*
2. *The restriction $N : \Gamma(V, \eta) \to \mathbb{R}^\times \cdot \mathrm{id}$ is a group homomorphism.*
3. *If $x \in \Gamma(V, \eta)$, then $N(\alpha(x)) = N(x)$.*

Proof. We check 1. If $x \in \Gamma(V, \eta)$, then $\tau(\widetilde{\mathrm{Ad}}(x)v) = \widetilde{\mathrm{Ad}}(x)v$ for all $v \in V$. On the otherhand

$$\tau(\widetilde{\mathrm{Ad}}(x)v) = \tau(\alpha(x)vx^{-1}) = \tau(x^{-1})v\kappa(x) = \widetilde{\mathrm{Ad}}(\kappa(x)^{-1})v \tag{1.41}$$

Thus, $\widetilde{\mathrm{Ad}}(x^{-1}\kappa(x^{-1}))v = \widetilde{\mathrm{Ad}}(N(x^{-1}))v = v$. So $N(x^{-1})$ is indeed in the kernel $\ker(\widetilde{\mathrm{Ad}}) = \mathbb{R}^\times \cdot \mathrm{id}$.

We check 2. Indeed,

$$N(xy) = xy\kappa(xy) = xN(y)\kappa(x) = N(x)N(y) \tag{1.42}$$

because $N(x) \in \mathbb{R}^\times \cdot \mathrm{id}$.

We check 3. Indeed,

$$N(\alpha(x)) = \alpha(x)\alpha(\kappa(x)) = \alpha(x\kappa(x)) = \alpha(N(x)) = N(x) \tag{1.43}$$

because $N(x) \in \mathbb{R}^\times \cdot \mathrm{id}$. □

Now, using these well-behaved properties of N, we can prove that the $\widetilde{\mathrm{Ad}}$-representation of $\Gamma(V, \eta)$ on V not only contains $O(V, \eta)$ in its image, but that each $\widetilde{\mathrm{Ad}}(x)$ is actually orthogonal.

Proposition 1.8. *If $x \in \Gamma(V,\eta)$, then $\widetilde{\mathrm{Ad}}(x) \in O(V,\eta)$ is orthogonal.*

Proof. We will prove that the norm $v \mapsto \eta(v) := \eta(v,v)$ is preserved and then use the polarisation identities to ensure the inner product is preserved.

Let $x \in \Gamma(V,\eta)$ and $v \in V$ such that $\eta(v) \neq 0$, then $v \in \Gamma(V,\eta)$. Moreover,

$$N(\widetilde{\mathrm{Ad}}(x)v) = N(\alpha(x)vx^{-1}) = N(x)N(v)N(x)^{-1} = N(v)\,. \tag{1.44}$$

Where, for any vector $w \in V$ $N(w) = -w^2 = \eta(w)$. Hence, the above shows $\eta(\widetilde{\mathrm{Ad}}(x)v) = \eta(v)$.

We will invoke topology to deal with the remaining non-zero null-vectors $v \in V$ with $\eta(v) = 0$. Such vectors have a unique expansion as $v = v_1 + v_2$ with $\eta(v_1) = -\eta(v_2) \neq 0$ where we use the usual decomposition $V = V_1 \oplus V_2$. Hence they can also be expressed as $v = \lim_{\varepsilon\to 0} v_\varepsilon$ where $v_\varepsilon = v_1 + (1+\varepsilon)v_2$ and $\eta(v_\varepsilon) = \varepsilon^2\eta(v_2) \neq 0$. Thus, we can write

$$\lim_{\varepsilon\to 0} N(\widetilde{\mathrm{Ad}}(x)v_\varepsilon) = \lim_{\varepsilon\to 0} N(v_\varepsilon)\,. \tag{1.45}$$

Now consider N restricted to V, as mentioned above, it is proportional to the quadratic form η, hence it is continuous. Moreover, linear maps (like $\widetilde{\mathrm{Ad}}(x)$) are continuous on V. Thus,

$$N(\widetilde{\mathrm{Ad}}(x)v) = N(v) \tag{1.46}$$

holds for null-vectors $v \neq 0$ with $\eta(v) = 0$. So the norms of all vectors in V are preserved. □

By putting the various bits of information together we have the short exact sequence

$$\boxed{1 \longrightarrow \mathbb{R}^\times \longrightarrow \Gamma(V,\eta) \xrightarrow{\widetilde{\mathrm{Ad}}} O(V,\eta) \longrightarrow 1\,.} \tag{1.47}$$

From here we can make an argument to explicitly characterise the Clifford–Lipschitz group $\Gamma(V,\eta)$.

Proposition 1.9. *The Clifford–Lipschitz group $\Gamma(V,\eta)$ can be equivalently defined as*

$$\Gamma(V,\eta) := \{\lambda v_1 \dots v_r \mid \eta(v_i) \neq 0\,, \lambda \in \mathbb{R}^\times\}\,. \tag{1.48}$$

In geometric algebra, this new redefinition is called the 'versor group'.

Proof. For now, let $\Gamma(V,\eta)$ take its usual definition and let $G := \{\lambda v_1 \dots v_r \mid \eta(v_i) \neq 0\,, \lambda \in \mathbb{R}^\times\}$. It is clear that $G \subset \Gamma(V,\eta)$.

Take $x \in \Gamma(V,\eta)$, then, because $\widetilde{\mathrm{Ad}}(x)$ is orthogonal, Cartan–Dieudonneé guarantees that there exists a product of reflections $y \in G$ such that $\widetilde{\mathrm{Ad}}(x) = \widetilde{\mathrm{Ad}}(y)$. Hence $xy^{-1} \in \ker \widetilde{\mathrm{Ad}} = \mathbb{R}^\times$. Thus, $x = \lambda y$ for some $\lambda \in \mathbb{R}^\times$. □

Finally we end with another characterisation of the Clifford–Lipschitz group $\Gamma(V,\eta)$.

Lemma 1.10. *The following are true*

1. *For any $x \in \mathrm{Cl}(V,\eta)$, if $\tau(x) = x$, then $x \in \mathbb{R} \oplus V$.*
2. *Let x be a unit. If $N(x) \in \mathbb{R}^\times \cdot \mathrm{id}$ is a non-zero scalar, then $\widetilde{\mathrm{Ad}}(x)v \in V$.*

Proof. We check 1. Indeed, fix a basis e_a of V which diagonalises $\eta_{ab} = \mathrm{diag}(1,\dots,-1)$. As τ is linear it suffices to look at its action on the basis $\{\mathrm{id}, e_{a_1}\dots e_{a_n}\}$ where $a_1 < \cdots < a_n$ and $n = 1,\dots,p+q$ of $\mathrm{Cl}(V,\eta)$. Clearly, only the basis vectors id, e_a are preserved. Hence, $\tau(x) = x$ implies $x \in \mathbb{R} \oplus V$.

We check 2. We have $x^{-1} = \lambda^{-1}\kappa(x)$ for some real $\lambda \neq 0$. Thus, any $v \in V$ obeys

$$\widetilde{\mathrm{Ad}}(x)v = \alpha(x)vx^{-1} = -\lambda^{-1}\alpha(xv\tau(x))\,. \tag{1.49}$$

Moreover

$$\tau(\widetilde{\mathrm{Ad}}(x)v) = -\lambda^{-1}\alpha(\tau(xv\tau(x))) = -\lambda^{-1}\alpha(xv\tau(x)) = \widetilde{\mathrm{Ad}}(x)v \tag{1.50}$$

Finally, $\widetilde{\mathrm{Ad}}(x)v$ must be in V by parity. □

As a result of the proposition and lemma we now have a nice theorem.

Theorem 1.11. *The following definitions are all equivalent*

1. $\Gamma(V,\eta) := \{x \in \mathrm{Cl}(V,\eta)^\times \mid \widetilde{\mathrm{Ad}}(x)(v) \in V \quad \text{for all } v \in V\}$.
2. $\Gamma(V,\eta) := \{\lambda v_1 \dots v_r \mid \eta(v_i) \neq 0\,, \lambda \in \mathbb{R}^\times\}$.
3. $\Gamma(V,\eta) := \{x \in \mathrm{Cl}(V,\eta)^\times \mid N(x) \in \mathbb{R}^\times \cdot \mathrm{id}\}$.

1.3.4 Pin and spin groups

We are now ready to define pin and spin groups. We will make an effort to relate to other definitions of them.

Definition 1.12. The *pin group* is defined as any of the equivalent definitions (by Theorem 1.11)

$$\mathrm{Pin}(V,\eta) := \{x \in \Gamma(V,\eta) \mid N(x) = \pm 1\}\,, \tag{1.51}$$
$$\mathrm{Pin}(V,\eta) := \{x \in \mathrm{Cl}(V,\eta)^\times \mid N(x) = \pm 1\}\,, \tag{1.52}$$
$$\mathrm{Pin}(V,\eta) := \{\pm v_1 \dots v_r \mid \eta(v_i) = \pm 1\}\,. \tag{1.53}$$

The *spin group* is defined as $\mathrm{Spin}(V,\eta) := \mathrm{Pin}^0(V,\eta)$, i.e. the even part. The *orthochronous spin group* is defined as $\mathrm{Spin}^+(V,\eta) := \{x \in \mathrm{Spin}(V,\eta) \mid N(x) = 1\}$. It also goes by the name 'rotor group' in geometric algebra.

General comments Note that $N(x) = \pm 1$ is indeed needed to define pin and spin groups in general. The plus or minus one is indicative of the presence of vectors v with negative norm $N(v) = -\eta(v) < 0$. This of course occurs for a non-degenerate indefinite quadratic form η. It also occurs for *negative-definite* quadratic forms. Indeed, in our $V = V_1 \oplus V_2$ convention with $\eta|_{V_1}$ positive-definite and $\eta|_{V_2}$ negative-definite, along with our Clifford algebra convention

$$vw + wv = -2\eta(v,w) \tag{1.54}$$

we find a unit time-like vector has $N(v_1) = -v_1^2 = \eta(v_1) = 1$ whereas a unit space-like vector has $N(v_2) = -v_2^2 = \eta(v_2) = -1$. Setting $V_1 = 0$ so that η is negative-definite then reveals that $N(v_2) = -1$ still occurs.

The only time when $N(x) = 1$ is sufficient to describe $\mathrm{Pin}(V,\eta)$ (in our conventions) is when $V = V_1$.

Setting $N(x) = 1$ is still useful though (as is evident in $\mathrm{Spin}^+(V,\eta)$'s definition). Indeed, in our conventions, setting $N(x) = 1$ forces $\widetilde{\mathrm{Ad}}(x)$ to always involve an *even number of time-like reflections*, hence temporal orientation is preserved and x is called *orthochronous*. As a consequence, if $\widetilde{\mathrm{Ad}}(x) \in O(V_1, \eta|_{V_1})$ only affects V_1, then $N(x) = 1$ iff $\det(\widetilde{\mathrm{Ad}}(x)) = 1$.

In particular, this means we have very explicit realisations of the groups as

$$\mathrm{Pin}(V,\eta) := \{\pm v_1 \dots v_r \mid \eta(v_i) = \pm 1\}\,, \tag{1.55a}$$
$$\mathrm{Spin}(V,\eta) := \{\pm v_1 \dots v_{2r} \mid \eta(v_i) = \pm 1\}\,, \tag{1.55b}$$
$$\mathrm{Spin}^+(V,\eta) := \{\pm v_1 \dots v_{2r} \mid \eta(v_i) = \pm 1, \text{ the number of } v_i \text{ such that } \eta(v_i) = 1 \text{ is even}\}\,. \tag{1.55c}$$

Definition 1.13. The *orthochronous special orthogonal group* is defined as $SO^+(V,\eta) := \widetilde{\mathrm{Ad}}(\mathrm{Spin}^+(V,\eta))$.

From the above explicit realisation of $\mathrm{Spin}^+(V,\eta)$, it is obvious that $SO^+(V,\eta)$ is a subgroup of $SO(V,\eta)$ generated by products of reflections with an even number of time-like reflections and an even number of space-like reflections.

From here onwards it is easier to state/read the results by using the signature (p,q) of (V,η), so we state the following in that notation.

Theorem 1.14. *The following results hold*

1. *The following are exact sequences*

$$1 \longrightarrow \{-1,1\} \longrightarrow \mathrm{Pin}(p,q) \xrightarrow{\widetilde{\mathrm{Ad}}} O(p,q) \longrightarrow 1\,, \tag{1.56}$$

$$1 \longrightarrow \{-1,1\} \longrightarrow \mathrm{Spin}(p,q) \xrightarrow{\widetilde{\mathrm{Ad}}} SO(p,q) \longrightarrow 1\,, \tag{1.57}$$

$$1 \longrightarrow \{-1,1\} \longrightarrow \mathrm{Spin}^+(p,q) \xrightarrow{\widetilde{\mathrm{Ad}}} SO^+(p,q) \longrightarrow 1\,, \tag{1.58}$$

   and $\widetilde{\mathrm{Ad}}$ *is a (double) covering map in each case.*

2. $\mathrm{Spin}^+(p,q) = \mathrm{Spin}(p,q)$ *iff* η *is positive-definite or negative definite. The same holds for* $SO^+(p,q) = SO(p,q)$.

3. $\mathrm{Spin}(p,q) \cong \mathrm{Spin}(q,p)$, *but not in general for pin groups.*

4. *The orthochronous special orthogonal group* $SO^+(p,q)$ *is connected. Hence, it is the identity connected component of the orthogonal groups.*

5. *The orthochronous spin group* $\mathrm{Spin}^+(p,q)$ *is connected except in the cases* $(p,q) = (1,1),(1,0),(0,1)$ *where instead* $\mathrm{Spin}^+(1,1) \cong \mathbb{R}^\times$, *and* $\mathrm{Spin}^+(1,0) = \mathrm{Spin}^+(0,1) = \{-1,1\}$. *Hence, it is (mostly) the identity connected component of the pin and spin groups and, in the cases where it is, all groups provide a non-trivial double cover of their respective orthogonal groups.*

6. *For definite signatures,* $\pi_1(SO(1)) = 1$, $\pi_1(SO(2)) \cong \mathbb{Z}$, $\pi_1(SO(n)) \cong \mathbb{Z}_2$ *for* $n \geq 3$. *Moreover, for indefinite signatures* (p,q) *with* $p \geq 1$ *and* $q \geq 1$

$$\pi_1(SO^+(p,q)) \cong \begin{cases} 1 & \text{if } (p,q) = (1,1)\,, \\ \mathbb{Z} & \text{if } (p,q) = (1,2)\,, \\ \mathbb{Z}_2 & \text{if } p = 1 \text{ and } q \geq 3\,, \\ \mathbb{Z}\times\mathbb{Z} & \text{if } (p,q) = (2,2)\,, \\ \mathbb{Z}\times\mathbb{Z}_2 & \text{if } p = 2 \text{ and } q \geq 3\,, \\ \mathbb{Z}_2\times\mathbb{Z}_2 & \text{if } p,q \geq 3\,, \end{cases} \tag{1.59}$$

   keeping in mind that $O(p,q) = O(q,p)$. *Consequently, due to the non-trivial double covering* $\pi_1(\mathrm{Spin}(2)) \cong \mathbb{Z}$, $\pi_1(\mathrm{Spin}(n)) \cong 1$ *for* $n \geq 3$. *For indefinite signatures*

$$\pi_1(\mathrm{Spin}^+(p,q)) \cong \begin{cases} 1 & \text{if } (p,q) = (1,1)\,, \\ \mathbb{Z} & \text{if } (p,q) = (1,2)\,, \\ 1 & \text{if } p = 1 \text{ and } q \geq 3\,, \\ ? & \text{if } (p,q) = (2,2)\,, \\ ? & \text{if } p = 2 \text{ and } q \geq 3\,, \\ ? & \text{if } p,q \geq 3\,, \end{cases} \tag{1.60}$$

   keeping in mind that $\mathrm{Spin}(p,q) = \mathrm{Spin}(q,p)$ *(?'s are for ones I am unsure of). For more details about the uniqueness of double covering see Varadarajan's book. The key point is that, in a good chunk of cases, the covering is not universal.*

Proof. We check 1.

Since $\ker \widetilde{\mathrm{Ad}} = \mathbb{R}^\times = N(\Gamma(p,q))$ for the Clifford–Lipschitz group, then it is clear that the kernel must restrict to $\ker \widetilde{\mathrm{Ad}} = \{-1, 1\} = \{\pm N(\mathrm{Pin}(p,q))\}$ for the Pin subgroup (note that $N(x) = N(-x)$). Similarly, for the spin and orthochronous spin subgroups. Hence, each point $p \in \mathrm{im}\, \widetilde{\mathrm{Ad}}$ has a preimage of 2 points $\{x, -x\}$ in the pin, spin and orthochronous spin groups. We skip checking this is a local homeomorphism, but it indeed is.

We check 2.

This is immediate from (1.55). Indeed, $\mathrm{Spin}^+(V, \eta)$ contains even products of unit vectors which also have an even number of unit time-like vectors. If there are only time-like vectors (positive-definite), then $\mathrm{Spin}^+(p, 0) = \mathrm{Spin}(p, 0)$ because $\mathrm{Spin}(p, 0)$ just contains even products of unit vectors. The same holds for the negative-definite case. If η is indefinite, i.e. p, q are both non-zero, then a product of a unit time-like and space-like vector is in $\mathrm{Spin}(p,q)$, but not $\mathrm{Spin}^+(p,q)$, so they are not equal. The special orthogonal version is the same argument, just with reflections (take the image under $\widetilde{\mathrm{Ad}}$).

We check 3.

This is easier to check later when we classify real Clifford algebras. In particular, we will find a non-canonical algebra isomorphism $\mathrm{Cl}^0(p,q) \cong \mathrm{Cl}^0(q,p)$ which descends to a group isomorphism $\mathrm{Spin}(p,q) \cong \mathrm{Spin}(q,p)$.

We check 4.

Fix a decomposition $V = V_1 \oplus V_2$, and note that $\sigma_v \sigma_w = \sigma_{\sigma_v(w)} \sigma_v$ where $\eta(w) = \eta(\sigma_v(w))$ preserves that w is space-like ($\eta(w) < 0$) or time-like ($\eta(w) > 0$). This ensures any product of reflections $\sigma_{v_1} \dots \sigma_{v_r}$ can be rewritten with time-like reflections grouped on the left and space-like reflections grouped on the right. Now, we can of course make a path between any two vectors v, w by

$$\gamma(t) = (1-t)v + tw\,. \tag{1.61}$$

However, if $\eta(v) > 0$ and $\eta(w) < 0$ intermediate value theorem dictates that $\eta(\gamma(t)) = 0$ for some t and hence the curve would *not* always have a well-defined reflection σ. Now, any time-like vector has a unique decomposition $v = v_1 + v_2$, $\eta(v) > 0$ and can be continuously joined to v_1 by

$$\gamma(t) = v_1 + (1-t)v_2 \tag{1.62}$$

where $\eta(\gamma(t)) = \eta(v_1) + (1-t)^2 \eta(v_2) \geq \eta(v) > 0$. Moreover, $\eta|_{V_1}$ is positive-definite, so any vectors within it may be connected by a path and retain positive norm. Hence, by continuity, there is a path $\gamma(t)$ in $O(p,q)$ between any two time-like reflections. The same arguments hold for space-like reflections. Using that $\sigma_v^2 = 1$ is idempotent for any $v \in V$ such that $v \neq 0$, it is clear that any element of $SO^+(p,q)$ is connected to the identity precisely because such elements have an even number of time-like reflections and space-like reflections.

We check 5.

In the two 1-dimensional cases, the definitions themselves (1.55) ensures $\mathrm{Spin}^+(1,0) = \mathrm{Spin}^+(0,1) = \{-1, 1\}$. In the $\mathrm{Spin}^+(1,1)$ case, fix an orthonormal basis $\{e_1, e_2\}$ of V with $\eta(e_1) = -\eta(e_2) = 1$. We claim that every element $x \in \mathrm{Spin}^+(1,1)$ is of the form

$$x = \pm e^{t e_1 e_2} = \pm(\cosh(t) + \sinh(t) e_1 e_2) \tag{1.63}$$

for some $t \in \mathbb{R}$. This is not hard to check as any element of $\mathrm{Cl}^0(1,1)$ has a unique expansion $x = a + b e_1 e_2$ and $N(x) = a^2 - b^2$. Hence, $x \in \mathrm{Spin}^+(1,1)$ iff $a^2 - b^2 = 1$. This has a general solution $x = \pm\sqrt{1 + b^2} + b e_1 e_2$ for $b \in \mathbb{R}$. The invertible relation $b = \sinh(t)$ gives the result. A group isomorphism $\mathrm{Spin}^+(1,1) \to \mathbb{R}^\times$ is then given by $\pm e^{t e_1 e_2} \mapsto \pm e^t$.

Suppose $\dim V = p + q \geq 2$ and $(p,q) \neq (1,1)$. We of course know that $SO^+(p,q)$ is connected and $\widetilde{\mathrm{Ad}} : \mathrm{Spin}^+(p,q) \to SO^+(p,q)$ is a double covering with $\ker \widetilde{\mathrm{Ad}} = \{-1, 1\}$. Hence $\mathrm{Spin}^+(p,q)$ is connected iff ± 1 live in the same connected component. Due to our (p,q) assumptions we can find orthonormal vectors e_1, e_2 on which η is positive-definite or negative definite. That is

$(\eta(e_1), \eta(e_2)) = (1,1), (-1,-1)$. W.l.o.g. take the $(1,1)$ case and define a path γ in $\mathrm{Spin}^+(p,q)$ by

$$\gamma(t) = (\cos te_1 + \sin te_2)(\cos te_1 - \sin te_2) \tag{1.64}$$

Clearly, $\gamma(0) = e_1^2 = -\eta(e_1) = -1$ and $\gamma(\frac{\pi}{2}) = -e_2^2 = \eta(e_2) = 1$ gives a path from -1 to 1. This also works for the $(-1,-1)$ case, so ± 1 are indeed connected. Note it would *fail* if we tried it for the $(1,-1)$ case as $\eta((\cos(\frac{\pi}{4})e_1 + \sin(\frac{\pi}{4})e_2)) = 0$ has zero norm.

We check 6.

We give a sketch. First $SO(1) = 1$ is just the trivial group with a single point given by the identity element, so $\pi_1(SO(1)) = 1$ is trivial. Second $SO(2) \cong U(1) \cong S^1$ is a 'nice' topological space[4] and it is universally covered by $\mathbb{R}$ with fiber $\mathbb{Z}$, so $\pi_1(SO(2)) \cong \mathbb{Z}$. In the case of $SO(3)$, one can check explicitly using that $\mathrm{Spin}(3) \cong SU(2) \cong S^3$. Then use that $\pi_1(S^3) = 1$ is trivial so that Spin(3) is a universal double cover, which means $\pi_1(SO(3)) \cong \mathbb{Z}_2$. To deal with $n \geq 3$ in general we use the fibration $SO(n) \to SO(n+1) \to S^n$[5] and its long exact sequence in homotopy groups. Indeed, the sequence

$$\pi_2(S^n) \longrightarrow \pi_1(SO(n)) \longrightarrow \pi_1(SO(n+1)) \longrightarrow \pi_1(S^n) \tag{1.65}$$

is exact and $\pi_1(S^n) = \pi_2(S^n) = 1$ are trivial since $n \geq 3$. Thus, $\pi_1(SO(n)) \cong \pi_1(SO(n+1))$ for $n \geq 3$.

In the indefinite case we use the fact that $SO^+(p,q)$ has a maximal compact subgroup given by $SO(p) \times SO(q)$ and that Cartan's decomposition theorem for Lie groups (Theorem 1.15) ensures a Lie group deformation retracts onto a maximal compact subgroup. Thus, we get the rest from

$$\pi_1(SO^+(p,q)) \cong \pi_1(SO(p)) \times \pi_1(SO(q))\,. \tag{1.66}$$

□

In the above proof have made use of the following theorem so that Lie groups deformation retract onto maximal compact subgroups.

Theorem 1.15 (Cartan)**.** *A connected real Lie group G is diffeomorphic to $K \times \mathbb{R}^n$ where K is a maximal compact subgroup of G. Moreover, all maximal compact subgroups are conjugate to K.*

1.3.5 A brief look at representation theory—from complex simple Lie algebra p.o.v.

We look at only *finite dimensional complex* representations here. We give a quick rundown of some rep theory results on semisimple Lie algebras.

The classical complex Lie algebras come in the four families

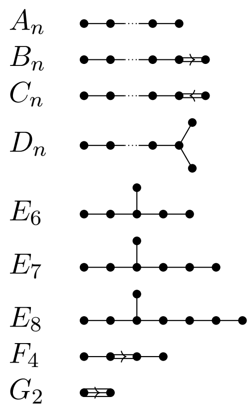
$$A_n := \mathfrak{sl}(n+1, \mathbb{C})\,, \tag{1.67}$$
$$B_n := \mathfrak{so}(2n+1, \mathbb{C})\,, \tag{1.68}$$
$$C_n := \mathfrak{sp}(2n, \mathbb{C})\,, \tag{1.69}$$
$$D_n := \mathfrak{so}(2n, \mathbb{C})\,, \tag{1.70}$$

and, along with the exceptional algebras E_6, E_7, E_8, F_4, G_2, they list all the complex simple Lie algebras (up to low-dimensional cases $D_1 = \mathfrak{so}(2,\mathbb{C}), D_2 = \mathfrak{so}(4,\mathbb{C})$, which are not simple). They

[4] This has to do with path-connected, locally path-connected, semi-locally path connected etc. Such requirements are fulfilled by connected manifolds (like the circle S^1).

[5] This is in fact a principal bundle with structure group $SO(n)$. One can realise this fibration easily with $SO(n+1) \to S^n \subseteq \mathbb{R}^{n+1}$ by $B \mapsto Be_{n+1}$ so that a preimage of any point is diffeomorphic to $SO(n)$.

have Dynkin diagrams



which correspond to a graphical realisation of the simple roots and their relations via the Cartan matrix $A = (a_{ij})$—a node for each simple root and an edge between nodes based on entries in the Cartan matrix. A key point is that the *fundamental weights* ω_i are related to the simple roots by α_i by $\alpha_i = a_{ij}\omega_j$. Every weight λ corresponds to irrep with highest weight λ and *all* irreps arise this way. The fundamental weights have the property that *any* weight λ is a unique non-negative integer linear combination of them.

Noting that, given two highest weight irreps $V_{\lambda_1}, V_{\lambda_2}$, the tensor product $V_{\lambda_1} \oplus V_{\lambda_2}$ contains the irrep $V_{\lambda_1+\lambda_2}$, we can see that all irreps can be realised as subreps of a tensor product of the fundamental reps V_{ω_i}. Finally, Weyl's theorem on complete reducibility ensures that all reps can be decomposed into a sum of irreps, so that we can construct *all* representations using only V_{ω_i}.

Theorem 1.16 (Weyl's theorem on complete reducibility)**.** *Consider a semisimple Lie algebra over a field k of characteristic zero. All its finite dimensional reps over k are completely reducible.*

Proof. Use Whitehead's first lemma and follow Wiki. Alternatively, over $k = \mathbb{C}$, one can leverage compact real forms in the spirit of Weyl. □

This theory holds generally for complex semisimple Lie algebras (which are direct sums of the simple ones). However, we are interested in the case of the special orthogonal algebras $\mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C}), \mathfrak{so}(2n, \mathbb{C}) = B_n, D_n$. Due to the double covering and connectedness results, we find that all the following real Lie algebras are the same

$$\mathfrak{pin}(p,q) \cong \mathfrak{spin}(p,q) \cong \mathfrak{spin}^+(p,q) \cong \mathfrak{so}(p,q) \tag{1.71}$$

Moreover, we can complexify to get $\mathfrak{so}(p,q) \otimes \mathbb{C} \cong \mathfrak{so}(p+q, \mathbb{C})$. The simple result below then determines the structure of complex representations of $\mathfrak{so}(p,q)$.

Proposition 1.17. *Let $\mathfrak{g}$ be a real Lie algebra, then any complex representation V gives a complex representation of its complexification $\mathfrak{g}_\mathbb{C}$. Moreover, by restriction we can take a complex representation of $\mathfrak{g}_\mathbb{C}$ to one for $\mathfrak{g}$. These operations are inverse.*

Proof. Given a complex representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ we need only define

$$\rho_\mathbb{C}(x+iy) := \rho(x) + i\rho(y) \tag{1.72}$$

for each $x, y \in \mathfrak{g}$ to lift it to a representation of $\mathfrak{g}_\mathbb{C}$ because ρ is $\mathbb{R}$-linear and already preserves $\mathfrak{g}$'s Lie bracket and

$$[x+iy, z+iw]_\mathbb{C} := [x,y] - [y,w] + i([x,w] + [y,z])\,. \tag{1.73}$$

Given a complex representation of $\mathfrak{g}_\mathbb{C}$ we simply restrict to $\mathfrak{g} \subseteq \mathfrak{g}_\mathbb{C}$ and $\mathbb{C}$-linearity and $[-,-]_\mathbb{C}$ preservation immediately restrict to $\mathbb{R}$-linearity and $[-,-]$ preservation.

These operations obviously undo eachother, hence are inverse. □

The key point of all of this is that the fundamental rep(s) corresponding to the right-most nodes on the Dynkin diagrams B_n, D_n are the *fundamental spinor rep(s)*, i.e. the complex spin-$\frac{1}{2}$ representation(s) of $\mathfrak{so}(p+q,\mathbb{C})$ and $\mathfrak{so}(p,q)$. In physics these would be called the *complex Weyl spinor rep(s)*.

Moreover, these spinor rep(s) can be realised as representations of the spin group $\mathrm{Spin}^+(p,q)$, even in the cases where it is *not* simply connected. This could be seen as being due to the complexification $\mathrm{Spin}^+(p+q,\mathbb{C})$ being simply connected, and all of complex rep theory depending only on complexified versions of groups/algebras.[6] In case this is unclear, we are referencing Lie's three theorems relating Lie algebras and Lie groups, in particular that a Lie algebra is (in some precise way) the same information as a connected, simply connected Lie group.

The non-spinor fundamental representations all arise in the following way (from Fulton-Harris section 19.2)

Theorem 1.18. *The following hold for $n \geq 0$ (maybe issues with $D_1 = \mathfrak{so}(2,\mathbb{C}), D_2 = \mathfrak{so}(4,\mathbb{C})$):*

1. *Let $V = \mathbb{C}^{2n}$ be the defining/standard representation of $\mathfrak{so}(2n,\mathbb{C})$. All non-spinor fundamental representations are $V, \Lambda^2 V, \ldots, \Lambda^{n-2}V$. There are two remaining fundamental spinor representations.*

2. *Let $V = \mathbb{C}^{2n+1}$ be the defining/standard representation of $\mathfrak{so}(2n+1,\mathbb{C})$. All non-spinor fundamental representations are $V, \Lambda^2 V, \ldots, \Lambda^{n-1}V$. There is one remaining fundamental spinor representation.*

1.4 Classification of real Clifford algebras

It is clear that representations of the algebra $\mathrm{Cl}(p,q)$ give representations of its group of units $\mathrm{Cl}(p,q)^\times$ and hence all of its subgroups, like $\mathrm{Pin}(p,q)$ by restriction. Similar things can be said about $\mathrm{Cl}^0(p,q)$ and the even subgroups.

We will find that the fundamental spinor representation(s), i.e. the complex spin-$\frac{1}{2}$ representation(s) of $\mathrm{Spin}^+(p,q)$ will arise as the only irreps of $\mathrm{Cl}^0(p,q)$.

The first step to this result is to classify the Clifford algebras as being matrix algebras or the sum of two matrix algebras. First we observe some low-dimensional examples.

Proposition 1.19. *The following low-dimensional isomorphisms hold:*

1. $\mathrm{Cl}(1,0) \cong \mathbb{C}$,

2. $\mathrm{Cl}(0,1) \cong \mathbb{R} \oplus \mathbb{R}$,

3. $\mathrm{Cl}(2,0) \cong \mathbb{H}$,

4. $\mathrm{Cl}(0,2) \cong \mathrm{Mat}_2(\mathbb{R})$,

5. $\mathrm{Cl}(1,1) \cong \mathrm{Mat}_2(\mathbb{R})$.

Proof. We check 1. This follows from writing $\mathrm{Cl}(1,0)$ w.r.t. the usual basis $\{\mathrm{id}, e_1\}$ with $(e_1)^2 = -\eta(e_1) = -1$.

We check 2. Similar, but now we have basis $\{\mathrm{id}, e_1\}$ with $(e_1)^2 = -\eta(e_1) = 1$. Hence, as a vector space we have $\mathbb{R}^2 = \mathbb{R} \oplus \mathbb{R}$ and then we put some multiplication on it.

We check 3. In this case we have a basis $\{\mathrm{id}, e_1, e_2, e_1e_2\}$ where $(e_1)^2 = (e_2)^2 = (e_1e_2)^2 = -1$. Hence, a map to $\mathbb{H}$ is given by $e_1 \mapsto i$, $e_2 \mapsto j$, $e_1e_2 \mapsto ij = k$.

[6] One defines the complex spin groups much the same way as we define the real ones. Just follow the earlier processes, but for $k = \mathbb{C}$.

We check 4. In this case we have a basis $\{\mathrm{id}, e_1, e_2, e_1e_2\}$ where $(e_1)^2 = (e_2)^2 = 1$ and $(e_1e_2)^2 = -1$. In this case consider the map

$$\mathrm{id} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad e_1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e_2 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad e_1e_2 \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{1.74}$$

Such matrices clearly span $\mathrm{Mat}_2(\mathbb{R})$.

We check 5. In this case we have a basis $\{\mathrm{id}, e_1, e_2, e_1e_2\}$ with $(e_1)^2 = -1$ and $(e_2)^2 = (e_1e_2)^2 = 1$. In this case we almost copy $\mathrm{Cl}(0,2)$ and use the map

$$\mathrm{id} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad e_1 \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad e_2 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad e_1e_2 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.75}$$

□

Next we note that larger Clifford algebras can be constructed from smaller ones

Lemma 1.20. *The following 'periods' hold:*

1. $\mathrm{Cl}(d+2,0) \cong \mathrm{Cl}(0,d) \otimes \mathrm{Cl}(2,0)$,
2. $\mathrm{Cl}(0,d+2) \cong \mathrm{Cl}(d,0) \otimes \mathrm{Cl}(0,2)$,
3. $\mathrm{Cl}(p+1,q+1) \cong \mathrm{Cl}(p,q) \otimes \mathrm{Cl}(1,1)$.

Proof. This is easiest to see at the level of generator mappings. Moreover, I find the 'gamma matrix' notation nicer here.

We check 1. Let $\Gamma'_1, \ldots, \Gamma'_d$ be the generators of $\mathrm{Cl}(0,d)$ and Γ''_1, Γ''_2 be the generators of $\mathrm{Cl}(2,0)$. Define a map $\mathrm{Cl}(0,d) \otimes \mathrm{Cl}(2,0) \to \mathrm{Cl}(d+2,0)$ by its value on the generators Γ_a of $\mathrm{Cl}(d+2,0)$ as

$$\Gamma_a = \begin{cases} \Gamma'_a \otimes \Gamma''_1\Gamma''_2 \,, \text{for } 1 \le a \le d \,, \\ \mathrm{id} \otimes \Gamma''_1 \,, \text{for } a = d+1 \,, \\ \mathrm{id} \otimes \Gamma''_2 \,, \text{for } a = d+2 \,. \end{cases} \tag{1.76}$$

Indeed, □

1.5 Clifford algebra (almost) as a group algebra

A key point about finite groups G is that their representations over a field k are in correspondence (bijection) with representations of the k-group algebra $k[G] = \mathrm{span}(e_g)_{g \in G}$.

A further point is if $k_1 \subset k_2$ is a subfield of k_2 then k_2-representations of G are in correspondence with k_2-representations of $k_1[G]$. So we may take $k_1 = \mathbb{R}$, $k_2 = \mathbb{C}$ if we want.

Gamma group. One can *almost* redefine $\mathrm{Cl}(p,q)$ as the real group algebra generated by the finite group G_Γ defined by the generators $\{\mathrm{id}, -1, \Gamma_a\}$[7] obeying the relations

- $(-1)^2 = \mathrm{id}$ and $(-1)\Gamma_a = \Gamma_a(-1)$,
- $(\Gamma_a)^2 = -1$ if $a = 0, \ldots, p-1$,
- $(\Gamma_a)^2 = \mathrm{id}$ if $a = p, \ldots, p+q-1$,
- $\Gamma_a\Gamma_b = (-1)\Gamma_b\Gamma_a$ if $a \neq b$.

[7] Following the wikipedia page on higher-dimensional gamma matrices (almost).

Due to these relations we find all elements $g \in G_\Gamma$ can be written uniquely as

$$g = (-1)^n \Gamma_{a_1} \dots \Gamma_{a_m} \tag{1.77}$$

where $n = 0, 1$ and $a_1 < \cdots < a_m$ and $0 \leq m \leq p + q$. Thus this is indeed a finite group and has $|G_\Gamma| = 2^{p+q+1}$ elements since

$$2^N = \sum_{i=0}^{N} \binom{N}{i} . \tag{1.78}$$

Group algebra and quotient. Indeed, the group algebra $k[G]$ over a field k of some group G has basis $\{e_g\}$. Thus we have the issue that e_{-1} is considered a distinct element. This is easy to fix, we just quotient by the ideal $I = \langle e_{-1} + e_{\mathrm{id}} \rangle$, i.e. set $e_{-1} = -e_{\mathrm{id}}$, and get

$$\mathbb{R}[G_\Gamma]/I = \mathrm{Cl}(p, q) . \tag{1.79}$$

Representations. We only consider representations $\rho : G_\Gamma \to GL(V)$ such that $\rho(\alpha) = -\rho(\mathrm{id})$. Such representations obviously extend to representations of $\mathbb{R}[G_\Gamma]/\langle e_\alpha = -e_{\mathrm{id}} \rangle = \mathrm{Cl}(p, q)$ and vice-versa.

The main point to all of this is that we can leverage some useful results of the representation theory of finite groups.

Lemma 1.21. *Given a complex representation V of a finite group G, there exists a G-invariant inner product on V.*

Proof. Take any inner product $\langle -, - \rangle$ on V, then a G-invariant inner product $\langle -, - \rangle_G$ is given by

$$\langle v, w \rangle_G := \sum_{g \in G} \langle gv, gw \rangle \tag{1.80}$$

where $gv := \rho(g)v$ is shorthand. □

As a consequence we get Maschke's theorem so that representations V always decompose into the direct sum of irreps.