Weighted Derangements And The Linearization Coefficients Of Orthogonal Sheffer Polynomials

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Jiang Zeng
Claude Bernard University Lyon 1

WEIGHTED DERANGEMENTS AND THE LINEARIZATION COEFFICIENTS OF ORTHOGONAL SHEFFER POLYNOMIALS

JIANG ZENG

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ABSTRACT

The present paper is devoted to a systematic study of the combinatorial interpretations of the moments and the linearization coefficients of the orthogonal Sheffer polynomials, i.e., Hermite, Charlier, Laguerre, Meixner and Meixner–Pollaczek polynomials. In particular, we show that Viennot's combinatorial interpretations of the moments can be derived directly from their classical analytical expressions and that the linearization coefficients of Meixner–Pollaczek polynomials have an interpretation in the model of derangements analogous to those of Laguerre and Meixner polynomials.

1. *Introduction*

Let $\{p_n(x)\colon n=0,1,2,\ldots\}$ be a sequence of polynomials, orthogonal on the real line with respect to a positive measure $d\alpha(x)$. One of the questions concerning the polynomials $p_n(x)$ is to determine, or at least say something useful about, the coefficients $a(k,n,m)$ in the expansion of

$$p_n(x)p_m(x)=\sum_{k=0}^{n+m} a(k,n,m)p_k(x). \tag{1.0}$$

This is usually called the linearization problem in the literature [**4**]. Taking orthogonality into account, we then have the equivalent equation:

$$\int_{-\infty}^{+\infty} p_n(x)p_m(x)p_k(x)\,d\alpha(x)=a(k,n,m)\int_{-\infty}^{+\infty} p_k(x)p_k(x)\,d\alpha(x).$$

The problem thus turns out to deal with the integrals of a product of three polynomials. More generally, we can define the *linearization coefficients* of $p_n(x)$, abbreviated by l.c. in what follows, to be the following integral of a product of m arbitrary polynomials:

$$\int_{-\infty}^{+\infty}\prod_{i=1}^{m} p_{n_i}(x)\,d\alpha(x). \tag{1.1}$$

Any such integral can, of course, be evaluated in finite terms by inserting the polynomial expressions for $p_{n_i}(x)$ and integrating term by term. However, as pointed out by several authors [**4, 20, 37**], the interesting problem is not just any representation of (1.1) but the one in which the non-negativity of the integral is apparent for some range of values of the parameters. Although it is well-known that the integral (1.1) is positive for most classical orthogonal polynomials [**3**], there is no simple general expression for (1.1). In the past two decades, many authors have also considered the refined problem of finding combinatorial interpretations of (1.1) for classical orthogonal polynomials. (See, for example,

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[**6, 7, 8, 14, 17, 20, 29, 22, 40, 41**].) In view of their analytical aspects, it should be interesting to find a geometric set-up of which the generating polynomial with *positive weight* is the integral (1.1) and which reduces to orthogonality when $m = 2$. The main object of the present paper is to show that such a combinatorial interpretation of (1.1) exists for all the *orthogonal Sheffer polynomials*.

Recall that a sequence of polynomials $p_1(x)$, $p_2(x)$, ... is said to be of Sheffer type if and only if the exponential generating function of these polynomials has the form

$$\sum_{n\geqslant 0} p_n(x)\frac{t^n}{n!} = f(t)\exp(xg(t)), \tag{1.2}$$

where $f(t)$, $g(t)$ are formal power series with $f(0) = 1$, $g(0) = 0$ and $g'(0) \neq 0$. As proved by Meixner [**12, 34**], there are only five classes of orthogonal Sheffer polynomials, i.e., the Hermite, Laguerre, Charlier, Meixner and Meixner–Pollaczek polynomials. To be more precise, we now give the explicit generating functions and weight functions of these polynomials. Throughout this paper, we denote by $(x)_n$ the shifted factorial $(x)_n = x(x+1)\ldots(x+n-1)$.

(a) Hermite polynomials $(H_n(x))_{n\geqslant 0}$:

$$\sum_{n\geqslant 0} H_n(x)\frac{t^n}{n!} = \exp(2xt - t^2), \tag{1.3}$$

orthogonal on $\mathbb{R}$ with respect to the weight function $x \mapsto e^{-x^2}$.

(b) Charlier polynomials $(C_n(x\,;a))_{n\geqslant 0}$:

$$\sum_{n\geqslant 0} C_n(x\,;a)\frac{t^n}{n!} = e^t\left(1-\frac{t}{a}\right)^x, \tag{1.4}$$

orthogonal on $\mathbb{Z}_+$ with respect to the weight function $x \mapsto a^x/x!$.

(c) Laguerre polynomials $(L_n^{(\alpha)}(x))_{n\geqslant 0}$:

$$\sum_{n\geqslant 0} L_n^{(\alpha)}(x)\frac{t^n}{n!} = (1-t)^{-\alpha-1}\exp\left(\frac{xt}{1-t}\right), \tag{1.5}$$

orthogonal on $\mathbb{R}_+$ with respect to the weight function $x \mapsto x^\alpha e^{-x}$.

(d) Meixner polynomials $(M_n(x\,;\beta, c))_{n\geqslant 0}$:

$$\sum_{n\geqslant 0} M_n(x\,;\beta, c)\frac{t^n}{n!} = \left(1-\frac{t}{c}\right)^x (1-t)^{-x-\beta}, \tag{1.6}$$

orthogonal on $\mathbb{Z}_+$ with respect to the weight function $x \mapsto (\beta)_x c^x/x!$.

(e) Meixner–Pollaczek polynomials $(P_n(x\,;\delta, \eta))_{n\geqslant 0}$:

$$\sum_{n\geqslant 0} P_n(x\,;\delta, \eta)\frac{t^n}{n!} = [(1+\delta t)^2 + t^2]^{-\eta/2}\exp\left[x\arctan\left(\frac{t}{1+\delta t}\right)\right], \tag{1.7}$$

orthogonal on $\mathbb{R}$ with respect to the weight function (here a misprint in [**12, 40**] has been corrected)

$$x \mapsto [\Gamma(\tfrac{1}{2}\eta)]^{-2}\,|\Gamma(\tfrac{1}{2}(\eta + ix))|^2\exp(x\arctan\delta).$$

Let m be a positive integer and $\mathbf{n}$ be a sequence $\mathbf{n} = (n_1, \ldots, n_m)$ of m

non-negative integers. The linearization coefficients of the five sets of the Sheffer polynomials can then be defined, with appropriate normalizations, as follows:

$$\mathscr{H}(\mathbf{n})=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\left(\prod_{i=1}^{m}2^{-n_i/2}H_{n_i}(x)\right)e^{-x^2}\,dx; \tag{1.8}$$

$$\mathscr{C}(\mathbf{n}\,;a)=e^{-a}\sum_{x=0}^{\infty}\left(\prod_{i=1}^{m}(-a)^{n_i}C_{n_i}(x\,;a)\right)\frac{a^x}{x!}; \tag{1.9}$$

$$\mathscr{L}(\mathbf{n}\,;\alpha)=\frac{1}{\Gamma(\alpha+1)}\int_{0}^{+\infty}\left(\prod_{i=1}^{m}(-1)^{n_i}L_{n_i}^{(\alpha)}(x)\right)x^{\alpha}e^{-x}\,dx; \tag{1.10}$$

$$\mathscr{M}(\mathbf{n}\,;\beta,c)=(1-c)^{\beta}\sum_{x=0}^{\infty}\left(\prod_{i=1}^{m}(-c)^{n_i}M_{n_i}(x\,;\beta,c)\right)\frac{c^x(\beta)_x}{x!}; \tag{1.11}$$

$$\mathscr{P}(\mathbf{n}\,;\delta,\eta)=\frac{1}{\int_{-\infty}^{+\infty}w(x)\,dx}\int_{-\infty}^{+\infty}\left(\prod_{i=1}^{m}P_{n_i}(x\,;\delta,\eta)\right)w(x)\,dx, \tag{1.12}$$

where $w(x)=[\Gamma(\frac{1}{2}\eta)]^{-2}\,|\Gamma(\frac{1}{2}(\eta+ix))|^2\exp(x\arctan\delta)$.

Although the combinatorial aspects of the l.c. of simple Laguerre polynomials $L_n^{(0)}(x)$ were considered first (cf. [**17**, **6**, **29**, **10**]), the interpretation of the l.c. of the general Laguerre polynomials was found only recently by Foata and Zeilberger [**20**]. The interpretation of the l.c. of Hermite polynomials is due to Godsil [**26**] and Azor, Gillis and Victor [**8**]. The combinatorial interpretations of the l.c. of Charlier and Meixner polynomials were recently found by Zeng [**41**]. Thus, as far as the linearization problems of Sheffer polynomials are concerned, there is still one interesting question remaining: is there an analogous interpretation for the linearization coefficients of Meixner–Pollaczek polynomials? The object of this paper is to give an affirmative answer to this question, but we also present a new approach to the linearization problem, which is based on generating functions and the interpretation of the moments.

Various approaches have been used hitherto to investigate the combinatorial interpretations of the linearization coefficients. In the earliest paper [**17**], Even and Gillis showed, by identifying the recurrence relations, that the l.c. of the simple Laguerre polynomials counts the 'generalized derangements': rearrangements of a *multiset* of objects of different 'colours' with the property that each object goes to an object of a different colour. Shortly after this, Askey and Ismail [**6**] observed that this result can be derived more directly by identifying the generating function of the integral (1.10) ($\alpha=0$) with that of the number of derangements, which was evaluated by MacMahon's Master theorem. Similarly, by refining the counting of derangements, they also found a combinatorial interpretation of the l.c. of Meixner polynomials $M_n(x\,;c,\beta)$ for $\beta=1$ [**6**]. In addition, Carlitz [**10**] also gave a short proof of the result of Even and Gillis by carrying out the integral (1.10) explicitly, still in the case where $\alpha=0$, and then applying an inversion formula or the principle of inclusion-exclusion. One decisive step was made by Foata and Zeilberger [**20**] in order to interpret the general α of (1.10). Instead of multisets, they consider the permutations of a total ordered 'coloured set', which permits them to introduce the notion of *cycle*.

Based on this geometric set-up, Foata and Zeilberger **[20]** and Zeng **[41]** successfully generalized both the Askey–Ismail and Carlitz methods to figure out the interpretations of the l.c. of Laguerre, Meixner, Charlier and Krawchouk polynomials, by using, in particular, the *exponential formula* and the β-extension of the MacMahon Master theorem.

Another different approach was initiated by Jackson **[29]**. He observed that the result of Even and Gillis **[17]** can be naturally derived from interpretations of the *moments* and from the *rook polynomial* interpretations of the simple Laguerre polynomials **[38]**. This approach has the advantage of relating the three different notions naturally: the (rook) polynomials, the moments, and the linearization coefficients. Similarly, Godsil **[26]** has given an interpretation of the l.c. of Hermite polynomials, but instead of rook polynomials he used *matching polynomials*. This method was later developed further by De Sainte–Catherine and Viennot **[14]** to provide an interpretation of the l.c. of Tchebycheff polynomials of the second kind; they also found the explicit killing involution under the principle of inclusion-exclusion involved in the above three proofs. More recently, Gessel **[22]** produced a common generalization of rook and matching polynomials to give new proofs of the known interpretations of the l.c. of the Hermite, general Laguerre, and Charlier polynomials.

Our interpretation of the l.c. of the Meixner–Pollaczek polynomials is again based on the model of derangements. Indeed, we will show that the integral (1.12) counts the derangements according to three statistics: *excedances, decedance* and *cycles* (cf. Theorem 4). This weight was originally inspired by Viennot's combinatorial interpretations of the moments of the Sheffer polynomials **[40]**. Moreover, we will propose two new approaches to the interpretation of the linearisation coefficients of Sheffer polynomials based on a *segment decomposition* of derangements.

For other combinatorial and group-theoretical aspects of the orthogonal polynomials discussed in this paper, we refer the reader to **[9, 19, 31, 32, 39, 40]**.

This paper is organized as follows. In § 2, we first evaluate the generating functions of the moments of Sheffer polynomials and then give their combinatorial interpretations (cf. Theorem 1), which imply Viennot's combinatorial results about the moments of Sheffer polynomials. In § 3, we use the generating functions of the moments to evaluate the generating functions of the l.c., in particular, that of the l.c. of Meixner–Pollaczek polynomials (cf. Theorem 2). In § 4, we introduce the notion of c-derangements and study their properties. In § 5, we evaluate the generating functions of the c-derangements according to various weights (cf. Theorem 3). In § 6, we deduce the combinatorial interpretations of the l.c. of Sheffer polynomials from their generating functions (cf. Theorem 4) and show that in the case where $\delta = 0$ it has another interesting interpretation in the context of *general up-down permutations* (cf. Theorem 5).

2. *The moments and combinatorial preliminaries*

It seems convenient to consider the integral in (1.1) as a formal linear functional on the vector space of polynomials. Thus the functionals corresponding to the Hermite, Charlier, Laguerre, Meixner, and Meixner–Pollaczek polynomials can be defined by their values on the base $\{x^n\}_{n\geq 0}$ as follows:

$$\nu(x^n)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}(\sqrt{2}\,x)^n e^{-x^2}\,dx; \tag{2.1}$$

$$\mu(x^n)=e^{-a}\sum_{x=0}^{\infty}x^n\frac{a^x}{x!}; \tag{2.2}$$

$$\psi(x^n)=\frac{1}{\Gamma(\alpha+1)}\int_0^{+\infty}x^n x^{\alpha}e^{-x}\,dx; \tag{2.3}$$

$$\rho(x^n)=(1-c)^{\beta}\sum_{x=0}^{\infty}x^n\frac{c^x(\beta)_x}{x!}; \tag{2.4}$$

$$\varphi(x^n)=\frac{1}{\int_{-\infty}^{+\infty}w(x)\,dx}\int_{-\infty}^{+\infty}x^n w(x)\,dx, \tag{2.5}$$

where $w(x)=[\Gamma(\frac{1}{2}\eta)]^{-2}\,|\Gamma(\frac{1}{2}(\eta+ix))|^2\exp(x\arctan\delta)$. One notices that the above quantities are actually the *moments* of the corresponding polynomials.

LEMMA 1. *We have*

$$\int_{-\infty}^{\infty}x^n e^{-(\pi-2\phi)x}\,|\Gamma(a+ix)|^2\,dx=2^{-n}\pi\Gamma(2a)\frac{d^n}{d\phi^n}(2\sin\phi)^{-2a}. \tag{2.6}$$

Proof. According to Pollaczek **[35]**, we have

$$\int_{-\infty}^{\infty}e^{-(\pi-2\phi)x}\,|\Gamma(a+ix)|^2\,dx=\pi\Gamma(2a)(2\sin\phi)^{-2a}.$$

Differentiating the above formula n times and then dividing both sides by 2^n, we immediately obtain (2.6).

PROPOSITION 1. *The following generating functions hold:*

$$\sum_{n\geq 0}\nu(x^n)\frac{t^n}{n!}=\exp(\tfrac{1}{2}t^2); \tag{2.7}$$

$$\sum_{n\geq 0}\mu(x^n)\frac{t^n}{n!}=\exp(a(e^t-1)); \tag{2.8}$$

$$\sum_{n\geq 0}\psi(x^n)\frac{t^n}{n!}=(1-t)^{-\alpha-1}; \tag{2.9}$$

$$\sum_{n\geq 0}\rho(x^n)\frac{t^n}{n!}=\left(\frac{1-c}{1-ce^t}\right)^{\beta}; \tag{2.10}$$

$$\sum_{n\geq 0}\varphi(x^n)\frac{t^n}{n!}=(\cos t-\delta\sin t)^{-\eta}. \tag{2.11}$$

Proof. Formulae (2.8) and (2.10) can be derived directly by inserting the series expressions and exchanging the order of the summations. Next, we notice that

$v(x^{2n+1})=0$ and

$$v(x^{2n})=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}(\sqrt{2}\,x)^{2n}e^{-x^2}\,dx=1\,.\,3\,.\,\ldots\,.\,(2n-1),$$

$$\psi(x^n)=\frac{1}{\Gamma(\alpha+1)}\int_0^{+\infty}x^n x^\alpha e^{-x}\,dx=(\alpha+1)_n.$$

Formulae (2.7) and (2.9) then follow immediately. Finally, we will give two proofs of (2.11). The first one uses only the special case $n=0$ of Lemma 1. By definition (2.5), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\varphi(x^n)\frac{t^n}{n!}&=\frac{\int_{-\infty}^{+\infty}e^{xt}w(x)\,dx}{\int_{-\infty}^{+\infty}w(x)\,dx}\\&=\frac{\int_{-\infty}^{+\infty}|\Gamma(\frac{1}{2}(\eta+ix))|^2\,e^{x(t+\arctan\delta)}\,dx}{\int_{-\infty}^{+\infty}|\Gamma(\frac{1}{2}(\eta+ix))|^2\,e^{x\arctan\delta}\,dx}\\&=\frac{\int_{-\infty}^{+\infty}|\Gamma(\frac{1}{2}\eta+iy)|^2\,e^{y(2t+2\arctan\delta)}\,dy}{\int_{-\infty}^{+\infty}|\Gamma(\frac{1}{2}\eta+iy)|^2\,e^{2y\arctan\delta}\,dy}\\&=\left[\frac{\sin(\frac{1}{2}\pi+t+\arctan\delta)}{\sin(\frac{1}{2}\pi+\arctan\delta)}\right]^{-\eta}\\&=(\cos t-\delta\sin t)^{-\eta}.\end{aligned}$$

The second one uses Lemma 1 and Lagrange's inversion formula. Comparing definition (2.5) and Lemma 1, we see that $\varphi(x^n)$ is the evaluation of the function

$$(\sin\phi)^\eta\frac{d^n}{d\phi^n}(\sin\phi)^{-\eta}$$

at $\phi=\frac{1}{2}\pi+\arctan\delta$. Thanks to Lagrange's inversion formula, or just Taylor's formula [**36**], we have

$$\begin{aligned}\sum_{n=0}^{\infty}(\sin\phi)^\eta\frac{d^n}{d\phi^n}(\sin\phi)^{-\eta}\frac{t^n}{n!}&=(\sin\phi)^\eta[\sin(\phi+t)]^{-\eta}\\&=(\cos t+\cot\phi\sin t)^{-\eta}.\end{aligned}$$

Substituting $\phi=\frac{1}{2}\pi+\arctan\delta$ into the last line, we obtain (2.11).

In order to give the combinatorial interpretations of these moments, we first need to introduce some basic definitions and notation. Let $\mathbb{N}$ be the set of non-negative integers and let $[n]=\{1,\ldots,n\}$ for $n\in\mathbb{N}$; by convention $[0]=\varnothing$. Let $\mathcal{S}_n$ be the set of permutations of $[n]$. For $\sigma\in\mathcal{S}_n$, we say that σ has an

excedance at $i\in[n]$ if $\sigma(i)>i$,
decedance at $i\in[n]$ if $\sigma(i)<i$,
maximum at $i\in[n]$ if $\sigma^{-1}(i)<i>\sigma(i)$,

and denote by $\operatorname{exc}\sigma$, $\operatorname{dec}\sigma$, and $\max\sigma$ respectively the numbers of excedances, decedance and maxima of σ. A *cycle* of σ is a sequence $(a,\sigma(a),\ldots,\sigma^{k-1}(a))$ such that $\sigma^k(a)=a$ and $\sigma^l(a)\neq a$ for $1<l<k$. The *length* of this cycle is k. When $k=1$, we say that a is a *fixed point* of σ. Clearly, each permutation σ can be factorized as a product of disjoint cycles. We denote by $\operatorname{cyc}\sigma$ the number of

cycles of σ. A permutation σ is said to be *circular* if $\operatorname{cyc}\sigma = 1$. Furthermore, we call σ a *derangement* of $[n]$ if $\sigma(i) \neq i$ for all $i \in [n]$ or, equivalently, all the cycles of σ have lengths at least 2, and we denote by $\mathscr{D}_n$ the set of derangements of $[n]$. Finally, we call σ an *involution* if $\sigma^2 = \mathrm{id}$ and denote by $\mathscr{I}_n$ the set of *fixed point free* involutions on $[n]$.

For a finite set S, we denote by $|S|$ the *cardinal* of S. A *partition* of S is an assemblage of the disjoint subsets (or blocs) $\pi = \{E_1, E_2, \ldots, E_l\}$ such that $E_1 + \ldots + E_l = S$. Let $\operatorname{bloc}\pi$ be the number of blocs of π and $\mathscr{P}_n$ the set of partitions of $[n]$.

Throughout this paper, we adopt the following vector notation. For $\mathbf{n} = (n_1, \ldots, n_m)$, we define

$$\mathbf{n}! = n_1! \ldots n_m!, \quad \mathbf{x}^{\mathbf{n}} = x_1^{n_1} \ldots x^{n_m}.$$

If $f(\mathbf{x}) := f(x_1, \ldots, x_m)$ is a power series in $x_1, \ldots, x_m$, we denote by $[\mathbf{x}^{\mathbf{i}}]f(\mathbf{x})$ the coefficient of $\mathbf{x}^{\mathbf{i}}$ in $f(\mathbf{x})$. However, for convenience, the coefficient of the monomial $x_1 x_2 \ldots x_m$ in $f(\mathbf{x})$ will be denoted by $[\mathbf{x}]f(\mathbf{x})$.

LEMMA 2. *The number of circular permutations on $[n]$ with k maxima is*

$$\left[z^k \frac{x^n}{n!}\right] \log\{\cosh xy - y^{-1} \sinh xy\}^{-1},$$

where $y = (1-z)^{1/2}$.

Proof. A proof of Lemma 2 can be found in [**27**, p. 273]. Note that this result is usually attributed to Entringer [**16**]. In fact, this formula had been given implicitly by André [**1**] in 1895. See also Dumont [**15**] for some related results.

LEMMA 3. *The number of permutations on $[n]$ with k maxima and l cycles is*

$$\left[\gamma^k \eta^l \frac{x^n}{n!}\right] (\cosh xy - y^{-1} \sinh xy)^{-\eta},$$

where $y = \sqrt{(1-\gamma)}$.

Proof. This is obvious on combining the exponential formula [**18, 30**] and Lemma 2.

THEOREM 1. *We have*

$$\nu(x^n) = |\mathscr{I}_n|, \tag{2.12}$$

$$\mu(x^n) = \sum_{\pi \in \mathscr{P}_n} a^{\operatorname{bloc}\pi}, \tag{2.13}$$

$$\psi(x^n) = \sum_{\sigma \in \mathscr{S}_n} (\alpha+1)^{\operatorname{cyc}\sigma} = (\alpha+1)_n, \tag{2.14}$$

$$\rho(x^n) = (1-c)^{-n} \sum_{\sigma \in \mathscr{S}_n} c^{\operatorname{dec}\sigma} \beta^{\operatorname{cyc}\sigma}, \tag{2.15}$$

$$\varphi(x^n) = \delta^n \sum_{\sigma \in \mathscr{S}_n} (1 + 1/\delta^2)^{\max\sigma} \eta^{\operatorname{cyc}\sigma}. \tag{2.16}$$

Proof. Formula (2.12) is trivial, since the number of involutions without fixed point on $[2n+1]$ and $[2n]$ is respectively 0 and $1.3. \ldots .(2n-1)$. On the other hand, it is well known [**13**] that the number of permutations of $[n]$ with k cycles is $s(n, k)$, the *Stirling number of the first kind,* and the number of partitions of $[n]$ with k blocs is $S(n, k)$, the *Stirling number of the second kind.* Moreover, we have (cf. [**13**] or [**36**, p. 44])

$$\sum_{k=0}^{\infty} \sum_{k=0}^{n} S(n, k) a^k \frac{t^n}{n!} = \exp(a(e^t - 1)),$$

$$\sum_{k=0}^{\infty} \sum_{k=0}^{n} s(n, k) \beta^k \frac{t^n}{n!} = (1-t)^{-\beta}.$$

The formulae (2.13) and (2.14) follow then by comparing (2.8) and (2.9) with the above identities. Next, thanks to Foata's fundamental transformation [**18**], we know that $\sum_{\sigma \in \mathscr{S}_n} c^{\operatorname{dec} \sigma}$ is equal to the nth Eulerian polynomial [**38**], of which the exponential generating function is

$$\frac{1-c}{1-ce^{t(1-c)}}.$$

This and the exponential formula [**18, 30**] imply that

$$\sum_{n \geqslant 0} \Big(\sum_{\sigma \in \mathscr{S}_n} c^{\operatorname{dec} \sigma} \beta^{\operatorname{cyc} \sigma} \Big) \frac{t^n}{n!} = \left(\frac{1-c}{1-ce^{t(1-c)}} \right)^{\beta}.$$

Comparing the above identity with (2.10) yields (2.15). Finally, it follows from Lemma 3 that

$$\begin{aligned} \sum_{n \geqslant 0} \delta^n \sum_{\sigma \in S_n} (1 + 1/\delta^2)^{\max \sigma} \eta^{\operatorname{cyc} \sigma} \frac{t^n}{n!} &= \left(\cosh(it) - \frac{\delta}{i} \sinh(it) \right)^{-\eta} \\ &= (\cos t - \delta \sin t)^{-\eta}. \end{aligned}$$

We obtain (2.16) by comparing the above identity with (2.11).

REMARK. Theorem 1 actually gives Viennot's combinatorial definitions of the moments of Sheffer polynomials [**40**]. Note that Viennot's original interpretation of $\rho(x^n)$ (respectively $\varphi(x^n)$) uses the number of *descents* (respectively *pics*) and *saillants*; however, by applying Foata's fundamental transformation [**18**], it is easily seen that these statistics have the same distribution as (dec, cyc) (respectively (max, cyc)).

The following results are useful in the new derivations of the interpetation of the l.c. of Sheffer polynomials. Let

$$f_m(\gamma, \eta) = \sum_{\sigma \in \mathscr{D}} \gamma^{\operatorname{dec} \sigma} \eta^{\operatorname{cyc} \sigma}, \tag{2.17}$$

$$g_m(\gamma, \eta) = \sum_{\sigma \in \mathscr{D}_n} \gamma^{\max \sigma} \eta^{\operatorname{cyc} \sigma}. \tag{2.18}$$

PROPOSITION 2. *We have*

$$1+\sum_{m\geq 1} f_m(\gamma,\eta)\frac{x^m}{m!}=\left\{1-\sum_{k\geq 2}(\gamma+\gamma^2+\ldots+\gamma^{k-1})\frac{x^k}{k!}\right\}^{-\eta}, \tag{2.19}$$

$$1+\sum_{m\geq 1} g_m(\gamma,\eta)\frac{x^m}{m!}=\left\{1-\sum_{k\geq 2}\gamma\sum_{l\geq 0}\binom{k-1}{2l+1}(1-\gamma)^l\frac{x^k}{k!}\right\}^{-\eta}. \tag{2.20}$$

Proof. We just prove (2.20) here since (2.19) can be proved in the same manner; see also **[42]**. For $\pi\in\mathcal{D}_n$, the weight function $w(\pi)=\gamma^{\max\pi}\eta^{\operatorname{cyc}\pi}$ is *multiplicative* with respect to the cycle decomposition and each π does not contain any *fixed point* (or cycle of length 1); hence we have, by the exponential formula (cf. **[18]**) and Lemma 2 with $y=\sqrt{(1-\gamma)}$,

$$\begin{aligned}1+\sum_{m\geq 1} g_m(\gamma,\eta)\frac{x^m}{m!}&=\exp\{\eta\log(\cosh xy-y^{-1}\sinh xy)^{-1}-\eta x\}\\&=\left\{\frac{e^{(1+y)x}+e^{(1-y)x}}{2}-y^{-1}\frac{e^{(1+y)x}-e^{(1-y)x}}{2}\right\}^{-\eta}\\&=\left\{1-\sum_{k\geq 2}(1-y^2)\sum_{l\geq 0}\binom{k-1}{2l+1}y^{2l}\frac{x^k}{k!}\right\}^{-\eta}.\end{aligned}$$

The proof is completed by substituting $y^2=1-\gamma$ in the last line.

COROLLARY 1. *We have*

$$\sum_{\sigma\in\mathcal{D}_n}(\delta+i)^{\operatorname{dec}\sigma}(\delta-i)^{\operatorname{exc}\sigma}\eta^{\operatorname{cyc}\sigma}=\delta^n\sum_{\sigma\in\mathcal{D}_n}(1+1/\delta^2)^{\max\sigma}\eta^{\operatorname{cyc}\sigma}.$$

Proof. This follows from Proposition 2 by noting that $\operatorname{dec}\sigma+\operatorname{exc}\sigma=n$ for $\sigma\in\mathcal{D}_n$.

3. *The generating function of the linearization coefficients*

In what follows we shall denote by e_k the kth $(0\leq k\leq m)$ elementary symmetric polynomial of $x_1,\ldots,x_m$, that is,

$$\sum_{j=0}^m e_jx^j=\prod_{k=1}^m(1+x_kx). \tag{3.1}$$

By convention, we set $e_k=0$ if $k>m$.

LEMMA 4. *We have*

$$\tan(\arctan x_1+\ldots+\arctan x_m)=\frac{\sum_{n\geq 0}(-1)^ne_{2n+1}}{\sum_{n\geq 0}(-1)^ne_{2n}}. \tag{3.2}$$

Proof. This is routine, by induction on m.

LEMMA 5. *With $i=\sqrt{-1}$, the following identities hold:*

$$\left(\sum_{n\geq 0}(-1)^ne_{2n}\right)^2+\left(\sum_{n\geq 0}(-1)^ne_{2n+1}\right)^2=\prod_{k=1}^m(1+x_k^2); \tag{3.3}$$

$$\sum_{n\geq 0}(-1)^n e_{2n}=\frac{1}{2}\left\{\prod_{k=1}^{m}(1+ix_k)+\prod_{k=1}^{m}(1-ix_k)\right\}; \tag{3.4}$$

$$\sum_{n\geq 0}(-1)^n e_{2n+1}=\frac{1}{2i}\left\{\prod_{k=1}^{m}(1+ix_k)-\prod_{k=1}^{m}(1-ix_k)\right\}. \tag{3.5}$$

Proof. Substituting x by i and $-i$ in (3.1), we get

$$\sum_{n\geq 0}(-1)^n e_{2n}+i\sum_{n\geq 0}(-1)^n e_{2n+1}=\prod_{k=1}^{m}(1+ix_k), \tag{3.6}$$

$$\sum_{n\geq 0}(-1)^n e_{2n}-i\sum_{n\geq 0}(-1)^n e_{2n+1}=\prod_{k=1}^{m}(1-ix_k). \tag{3.7}$$

Formulae (3.4) and (3.5) are then straightforward. We obtain (3.3) by multiplying (3.6) and (3.7).

LEMMA 6. *For the linear functional φ of* (2.5), *the following identity holds:*

$$\varphi\left(\exp x\sum_{k=1}^{m}\arctan x_k\right)=\prod_{k=1}^{m}(1+x_k^2)^{\eta/2}\left\{\sum_{n\geq 0}(-1)^n e_{2n}-\delta\sum_{n\geq 0}(-1)^n e_{2n+1}\right\}^{-\eta}. \tag{3.8}$$

Proof. By virtue of (2.11) of Proposition 1, we have

$$\begin{aligned}\varphi\left(\exp x\sum_{k=1}^{m}\arctan x_k\right)&=\sum_{n\geq 0}\varphi(x^n)\frac{(\arctan x_1+\ldots+\arctan x_m)^n}{n!}\\&=\left\{\cos\left(\sum_{k=1}^{m}\arctan x_k\right)-\delta\sin\left(\sum_{k=1}^{m}\arctan x_k\right)\right\}^{-\eta}\\&=\left\{\frac{1-\delta\tan(\arctan x_1+\ldots+\arctan x_m)}{\sqrt{(1+\tan^2(\arctan x_1+\ldots+\arctan x_m))}}\right\}^{-\eta}.\end{aligned}$$

Substituting (3.2) into the last line, we get

$$\varphi\left(\exp x\sum_{k=1}^{m}\arctan x_k\right)=\left\{\frac{\sum_{n\geq 0}(-1)^n e_{2n}-\delta\sum_{n\geq 0}(-1)^n e_{2n+1}}{\sqrt{((\sum_{n\geq 0}(-1)^n e_{2n})^2+(\sum_{n\geq 0}(-1)^n e_{2n+1})^2)}}\right\}^{-\eta},$$

and then (3.8), in view of (3.3).

We are now in a position to compute the generating function of the linearization coefficients of all the Sheffer polynomials and express them in terms of the elementary symmetric functions of $x_1,\ldots,x_m$ with the help of the generating functions of the moments and the polynomials.

THEOREM 2. *We have*

$$\sum_{\mathbf{n}\geq 0}\mathcal{H}(\mathbf{n})\frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!}=e^{e_2}, \tag{3.9}$$

$$\sum_{\mathbf{n}\geq 0}\mathcal{C}(\mathbf{n}\,;a)\frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!}=e^{ae_2+\ldots+ae_m}, \tag{3.10}$$

$$\sum_{\mathbf{n}\geq 0}\mathcal{L}(\mathbf{n}\,;\alpha)\frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!}=(1-e_2-2e_3-\ldots-(m-1)e_m)^{-(\alpha+1)}, \tag{3.11}$$

$$(3.12)\quad \sum_{\mathbf{n}\geq 0} \mathcal{M}(\mathbf{n}\,;\beta, c)\frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!} = \left(1 - \sum_{k=2}^{m} (c + \ldots + c^{k-1})e_k\right)^{-\beta},$$

$$(3.13)\quad \sum_{\mathbf{n}\geq 0} \mathcal{P}(\mathbf{n}\,;\delta, \eta)\frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!} = \left\{1 - \sum_{k=2}^{m} (\delta^k + \delta^{k-2})\sum_{l\geq 0} \binom{k-1}{2l+1}\left(-\frac{1}{\delta^2}\right)^l e_k\right\}^{-\eta}.$$

Proof. Equations (3.9)–(3.12) are already known **[6, 41]** and also easily established using the generating functions of the polynomials themselves. So we give here only the proof of (3.13). According to (1.7) and (1.12), we have

$$\sum_{\mathbf{n}\geq 0} \mathcal{P}(\mathbf{n}\,;\delta, \eta)\frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!} = \varphi\left(\prod_{k=1}^{m} \sum_{n_k\geq 0} P_{n_k}(x\,;\delta, \eta)\frac{x_k^{n_k}}{n_k!}\right)$$
$$= \prod_{k=1}^{m} ((1+\delta x_k)^2 + x_k^2)^{-\eta/2}\varphi\left(\exp x\sum_{j=1}^{m} \arctan\frac{x_j}{1+\delta x_j}\right).$$

Now we apply Lemma 6 to the above identity and obtain

$$(3.14)\quad \sum_{\mathbf{n}\geq 0} \mathcal{P}(\mathbf{n}\,;\delta, \eta)\frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!} = \prod_{k=1}^{m} (1+\delta x_k)^{-\eta}\left\{\sum_{n\geq 0} (-1)^n\hat{e}_{2n} - \delta\sum_{n\geq 0} (-1)^n\hat{e}_{2n+1}\right\}^{-\eta},$$

where $\hat{e}_k$ ($k \geq 0$) are the elementary symmetric polynomials of the variables:

$$\frac{x_1}{1+\delta x_1}, \ldots, \frac{x_m}{1+\delta x_m}.$$

Next, it follows from (3.3) and (3.4) of Lemma 5 that

$$(3.15)\quad \sum_{n\geq 0} (-1)^n\hat{e}_{2n} = \frac{1}{2}\left\{\prod_{k=1}^{m} \frac{1+(\delta+i)x_k}{1+\delta x_k} + \prod_{k=1}^{m} \frac{1+(\delta-i)x_k}{1+\delta x_k}\right\},$$

$$(3.16)\quad \sum_{n\geq 0} (-1)^n\hat{e}_{2n+1} = \frac{1}{2i}\left\{\prod_{k=1}^{m} \frac{1+(\delta+i)x_k}{1+\delta x_k} - \prod_{k=1}^{m} \frac{1+(\delta-i)x_k}{1+\delta x_k}\right\}.$$

Finally, putting (3.15) and (3.16) into (3.14) we get

$$(3.17)\quad \sum_{\mathbf{n}} \mathcal{P}(\mathbf{n}\,;\delta, \eta)\frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!} = \left\{\frac{i-\delta}{2i}\prod_{k=1}^{m} (1+(\delta+i)x_k) + \frac{i+\delta}{2i}\prod_{k=1}^{m} (1+(\delta-i)x_k)\right\}^{-\eta}$$
$$= \left\{1 + \frac{1+\delta^2}{2i}\sum_{k=1}^{m} ((\delta-i)^{k-1} - (\delta+i)^{k-1})e_k\right\}^{-\eta},$$

which implies (3.13) after simplification.

If $m = 3$, the explicit formulae can be obtained from Theorem 2 for the linearization coefficients of the Sheffer polynomials.

COROLLARY. *Let $\mathbf{n} = (n_1, n_2, n_3)$. We have the following explicit formulae:*

$$(3.18)\quad \mathcal{H}(\mathbf{n}) = \begin{cases} \dfrac{n_1!\,n_2!\,n_3!}{(s-n_1)!\,(s-n_2)!\,(s-n_3)!}, & \text{if } n_1+n_2+n_3 = 2s, \\ 0, & \text{if } n_1+n_2+n_3 \text{ is odd;} \end{cases}$$

$$(3.19)\quad \mathscr{C}(\mathbf{n}\,;a) = \sum_{s\geq 0} \frac{n_1!\,n_2!\,n_3!\,a^s}{(s-n_1)!\,(s-n_2)!\,(s-n_3)!\,(n_1+n_2+n_3-2s)!};$$

$$(3.20)\qquad \mathscr{L}(\mathbf{n}\,;\alpha)=\sum_{s\geq 0}\frac{n_1!\,n_2!\,n_3!\,2^{n_1+n_2+n_3-2s}(\beta)_s}{(s-n_1)!\,(s-n_2)!\,(s-n_3)!\,(n_1+n_2+n_3-2s)!};$$

$$(3.21)\qquad \mathscr{M}(\mathbf{n}\,;\beta,c)=\sum_{s\geq 0}\frac{n_1!\,n_2!\,n_3!\,(1+c)^{n_1+n_2+n_3-2s}c^s(\beta)_s}{(s-n_1)!\,(s-n_2)!\,(s-n_3)!\,(n_1+n_2+n_3-2s)!};$$

$$(3.22)\qquad \mathscr{P}(\mathbf{n}\,;\delta,\eta)=\sum_{s\geq 0}\frac{n_1!\,n_2!\,n_3!\,(\eta)_s(1+1/\delta^2)^s\delta^{n_1+n_2+n_3}2^{n_1+n_2+n_3-2s}}{(s-n_1)!\,(s-n_2)!\,(s-n_3)!\,(n_1+n_2+n_3-2s)!}.$$

REMARK. Formulae (3.18)–(3.21) have appeared in [**3**, **5**], but, to the author's knowledge, formula (3.22) seems new. Setting $n_3=0$ in the above corollary, we then recover the orthogonalities of the Hermite, Charlier, Laguerre, Meixner, and Meixner–Pollaczek polynomials.

4. *The combinatorics of c-derangements*

Let us begin with some multianalogues of the notions and notations introduced in § 2. For each $i\in[m]$, we define the coloured sets $\mathbb{N}_i=i\times\mathbb{N}$ and $[n]_i=i\times[n]$. Let $a=(i,j)\in\mathbb{N}_i$; we call the first term i the colour of a and set $c(a)=i$. Let $\mathbb{N}_*=[m]\times\mathbb{N}$ denote the disjoint union $\mathbb{N}_1\cup\ldots\cup\mathbb{N}_m$. For $\mathbf{n}=(n_1,\ldots,n_m)$, we note also $[\mathbf{n}]=[n_1]_1\cup\ldots\cup[n_m]_m$. Clearly, the set $\mathbb{N}_*$ is a *total ordered set* with respect to the lexicographic order.

It is convenient to identify a permutation π of a set A with the *digraph* on A having an edge $a\to\pi(a)$ for each $a\in A$. We will refer to an edge beginning at a point of colour i and ending at a point of colour j as an '$i\to j$' edge, calling it *pure* if $i=j$ and *mixed* if $i\neq j$. A sequence of consecutive edges $a\to\pi(a)\to\pi^2(a)\to\ldots\to\pi^k(a)$ $(k\geq 0)$ is called a *path* of the graph. We say also that this is a $c(a)\to c(\pi(a))\to c(\pi^2(a))\to\ldots\to c(\pi^k(a))$ path when we just take account of the colours.

Let π be a permutation on $[\mathbf{n}]$. An element $a\in[\mathbf{n}]$ is said to be a *c-fixed point* of π if $c(a)=c(\pi(a))$. A permutation without c-fixed point is said to be a *c-derangement* on $[\mathbf{n}]$. We denote respectively by $\mathscr{D}_{\mathbf{n}}$ and $\mathscr{I}_{\mathbf{n}}$ the sets of c-derangements and of c-fixed point free involutions on $[\mathbf{n}]$. Similarly, we say that π has a *c-excedance* (respectively *c-decedance*) at $a\in[\mathbf{n}]$ if $c(\pi(a))>c(a)$ (respectively $c(\pi(a))<c(a)$) and denote by $\operatorname{exc}\pi$ (respectively $\operatorname{dec}\pi$) the number of excedances (respectively decedances) of π.

PROPOSITION 3. *Let* $D(\mathbf{n}\,;\gamma,\eta)=\sum\gamma^{\operatorname{exc}\pi}\eta^{\operatorname{cyc}\pi}$ $(\pi\in\mathscr{D}_{\mathbf{n}})$ *and let* $\mathbf{n}^*$ *be any rearrangement of* $\mathbf{n}=(n_1,\ldots,n_m)$. *Then*

$$(4.1)\qquad D(\mathbf{n}\,;\gamma,\eta)=D(\mathbf{n}^*\,;\gamma,\eta).$$

Proof. Without loss of generality, we may assume that $\mathbf{n}^*$ is a rearrangement of $\mathbf{n}$ by just exchanging n_i and n_{i+1}, where $1\leq i\leq m-1$. We will prove (4.1) by constructing an explicit bijection Φ: $\mathscr{D}_{\mathbf{n}}\to\mathscr{D}_{\mathbf{n}^*}$ such that $\operatorname{exc}\pi=\operatorname{exc}\Phi(\pi)$ and $\operatorname{cyc}\pi=\operatorname{cyc}\Phi(\pi)$ for all $\pi\in\mathscr{D}_{\mathbf{n}}$.

Consider any c-derangement π of $[\mathbf{n}]$ identified with its diagraph G. Recall that any vertex of π is characterized by a pair of natural numbers (i,j), where i $(1\leq i\leq m)$ is the number of the colour to which it belongs and j $(1\leq j\leq n_i)$ is its serial number in that colour. Since there is no pure edge in G, each vertex of colour i or $i+1$ is necessarily located in either a cycle of which the vertices are only of colour i and $i+1$ or one of the following paths:

(I) $r_1 \to i \to l_1$, or $r_2 \to (i+1) \to l_2$,

(II) $r_3 \to (i \to (i+1))^{k_1} \to l_3$, or $r_4 \to ((i+1) \to i)^{k_2} \to l_4$,

(III) $r_5 \to (i \to (i+1))^{k_3} \to i \to l_5$, or $r_6 \to ((i+1) \to i)^{k_4} \to (i+1) \to l_6$,

where the r_j, $l_j \notin \{i, i+1\}$ for $1 \leqslant j \leqslant 6$ denote the colours of the corresponding vertices and $k_l \geqslant 1$ for $1 \leqslant l \leqslant 4$. Now apply the following transformation Φ to G:

(1) exchange the serial numbers of each vertex of colour i (respectively $i+1$) with those of its image in each path of Type (II) and in each cycle consisting only of vertices of colour i and $i+1$;

(2) exchange the colours i and $i+1$ of the vertices in the paths of Types (I) and (III).

Call the new graph G'. It is then readily seen that G' is a graph of a c-derangement on $[\mathbf{n}^*]$. Denote the corresponding c-derangement by $\Phi(\pi)$. Clearly, the transformation Φ is a bijection between $\mathscr{D}_{\mathbf{n}}$ and $\mathscr{D}_{\mathbf{n}^*}$ with the property that $\operatorname{cyc} \pi = \operatorname{cyc} \Phi(\pi)$. On the other hand, each excedance of colour $i+1$ (respectively i) of π in a path of Type (II) or a cycle consisting only of colour i and $i+1$ is mapped to an excedance of $\Phi(\pi)$ with the same colour, and the number of excedances contained in each path of Type (I) or (II) is invariant with respect to this transformation. Therefore the total number of excedances does not change after this transformation, that is, $\operatorname{exc} \pi = \operatorname{exc} \Phi(\pi)$.

COROLLARY 3. *Let* $E(\mathbf{n}; \gamma, \eta) = \sum \gamma^{\operatorname{dec} \pi} \eta^{\operatorname{cyc} \pi}$ $(\pi \in \mathscr{D}_{\mathbf{n}})$ *and* $\mathbf{n}^*$ *be any rearrangement of* $\mathbf{n} = (n_1, \ldots, n_m)$. *Then* $D(\mathbf{n}; \gamma, \eta) = D(\mathbf{n}^*; \gamma, \eta)$.

Proof. This follows from the fact that $\operatorname{dec} \pi = n_1 + \ldots + n_m - \operatorname{exc} \pi$ for each $\pi \in \mathscr{D}_{\mathbf{n}}$.

Consider the graph of any c-derangement π of $\mathscr{D}_{\mathbf{n}}$. Take any cycle of π and call X^0 the highest ranking vertex of this cycle. Let $X^i = \pi(X^0)$ for $i \geqslant 1$. Then this cycle can be linearly presented as

$$X^0 \to X^1 \to \ldots \to X^{p-1} (\to X^p = X^0). \tag{4.2}$$

Since π is a c-derangement, it follows that $p \geqslant 2$ and $c(X^i) \neq c(X^{i+1})$ for $0 \leqslant i \leqslant p-1$.

A linear arrangement $w = a_1 a_2 \ldots a_n$ of the elements of a subset $A \subseteq \mathbb{N}_*$ is said to be a *wave arrangement* of A if no two adjacent elements of w have the same colour, that is, $c(a_i) \neq c(a_{i+1})$ for $1 \leqslant i \leqslant n-1$. A wave arrangement $w = a_1 a_2 \ldots a_n$ is called a *wave segment* if moreover $n \geqslant 2$ and $c(a_1) > c(a_i)$ for all $2 \leqslant i \leqslant n$. In what follows, we will identify $w = a_1 a_2 \ldots a_n$ with the sequence $a_1 \to a_2 \to \ldots \to a_n$ and call it a $c(a_1) \to c(a_2) \to \ldots \to c(a_n)$ arrangement or segment. It follows that the cycle (4.2) can be divided up into wave segments each of which begins with vertices of the same colour as X^0. For example, if the cycle (4.2) is

$$(7,6) \to (2,5) \to (3,7) \to (7,1) \to (2,4) \to (1,1) \to (4,3) \to (7,2) \to (2,1),$$

we have the segments

$$w_1 = (7, 6) \to (2, 5) \to (3, 7),$$
$$w_2 = (7, 1) \to (2, 4) \to (1, 1) \to (4, 3),$$
$$w_3 = (7, 2) \to (2, 1).$$

Thus, each cycle of π corresponds in this way to a set of wave segments and a circular permutation acting on it. Let C_i be the set of cycles of π of which the highest ranking vertices are of colour i $(2 \leqslant i \leqslant m)$. By the above decomposition of cycles, each class C_i $(2 \leqslant i \leqslant m)$ corresponds to a pair (Ω_i, σ_i) where Ω_i is a set of wave segments beginning with a vertex of colour i, and σ_i is a permutation acting on it. Taking these into account we can identify any c-derangement π with an $(m-1)$-tuple $((\Omega_2, \sigma_2), \ldots, (\Omega_m, \sigma_m))$, with the product (juxtaposition) of all the wave segments in $\Omega_2 \cup \ldots \cup \Omega_m$ being a rearrangement of $[\mathbf{n}]$ and the permutations σ_i $(2 \leqslant i \leqslant m)$ acting on Ω_i.

To determine the transfer of the weight through this correspondence, we are led to introduce the corresponding weight on the set of wave segments. Let $w = a_1 a_2 \ldots a_n$ be a wave segment. By abuse of language, we call a_i $(1 \leqslant i \leqslant n)$ *c-decedance* if $c(a_{i-1}) < c(a_i) > c(a_{i+1})$ with the convention that $a_0 = a_n$ and $a_{n+1} = a_1$. Denote by $\operatorname{dec} w$ the number of c-decedances of w and for a set of wave arrangements Ω, define $\operatorname{dec} \Omega = \sum_{w \in \Omega} \operatorname{dec} w$.

PROPOSITION 4. *The correspondence described above, that is,*

$$\pi \mapsto ((\Omega_2, \sigma_2), \ldots, (\Omega_m, \sigma_m)),$$

has the following properties:

$$\operatorname{dec} \pi = \sum_{i=2}^{m} \operatorname{dec} \Omega_i \quad \textit{and} \quad \operatorname{cyc} \pi = \sum_{i=2}^{m} \operatorname{cyc} \sigma_i.$$

Proof. Firstly, it is readily seen that $\operatorname{cyc} \pi = \sum_{i=2}^{m} \operatorname{cyc} \omega_i$. It then suffices to consider the case where π is a *circular c-derangement.* Let $\pi \in \mathcal{D}_{\mathbf{n}}$ be a circular derangement with the unique circle $x^0 \to x^1 \to \ldots \to x^p \to x^0$, where x^0 is the highest ranking vertex and $x^l = \pi^l(x^0)$ for $1 \leqslant l \leqslant p$. Let $\Omega = \{w_1, w_2, \ldots, w_k\}$ be the wave segment decomposition of π with

$$w_1 = a_0 a_1 \ldots a_{l_1 - 1},$$
$$w_2 = a_{l_1} a_{l_1 + 1} \ldots a_{l_2 - 1},$$
$$\vdots$$
$$w_k = a_{l_k} a_{l_k + 1} \ldots a_p,$$

where $a_l = x^l$ for $0 \leqslant l \leqslant p$ and $c(x^0) = c(x^{l_1}) = \ldots = c(x^{l_k})$. Then the associated circular permutation σ acting on Ω is $w_1 \to w_2 \to \ldots \to w_k \to w_1$. Clearly, an element x^l is a c-decedance of π if and only if a_l is a c-decedance of some w_i. Hence $\operatorname{dec} \pi = \sum_{i=1}^{k} \operatorname{dec} w_i$.

5. *The generating functions of c-derangements*

THEOREM 3. *The following identities hold:*

$$(5.1)\quad \sum_{\mathbf{n}\geq 0} |\mathscr{I}_{\mathbf{n}}| \frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!} = e^{e_2};$$

$$(5.2)\quad \sum_{\mathbf{n}\geq 0} \Big(\sum_{\pi\in\mathscr{P}_{\mathbf{n}}} a^{\operatorname{bloc}\pi}\Big) \frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!} = e^{ae_2+\ldots+ae_m};$$

$$(5.3)\quad \sum_{\mathbf{n}\geq 0} \Big(\sum_{\pi\in\mathscr{D}_{\mathbf{n}}} \beta^{\operatorname{cyc}\pi}\Big) \frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!} = \Big(1 - \sum_{k=2}^{m} (k-1)e_k\Big)^{-\beta};$$

$$(5.4)\quad \sum_{\mathbf{n}\geq 0} \Big(\sum_{\pi\in\mathscr{D}_{\mathbf{n}}} c^{\operatorname{dec}\pi}\beta^{\operatorname{cyc}\pi}\Big) \frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!} = \Big(1 - \sum_{k=2}^{m} (c+\ldots+c^{k-1})e_k\Big)^{-\beta};$$

$$(5.5)\quad \sum_{\mathbf{n}\geq 0} \Big(\sum_{\pi\in\mathscr{D}_{\mathbf{n}}} (\delta-i)^{\operatorname{exc}\pi}(\delta+i)^{\operatorname{dec}\pi}\eta^{\operatorname{cyc}\pi}\Big) \frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!}$$
$$= \Big\{1 - \sum_{k=2}^{m} (\delta^k+\delta^{k-2}) \sum_{l\geq 0} \binom{k-1}{2l+1}\Big(\frac{-1}{\delta^2}\Big)^l e_k\Big\}^{-\eta}.$$

Proof. Equations (5.1)–(5.4) are already known [**20**, **28**, **40**]. In fact, formulae (5.1) and (5.2) are straightforward consequences of the *multivariable exponential formula* [**30**], and formulae (5.3) and (5.4) were proved in [**20**, **41**] by applying the β-extension of MacMahon's Master Theorem. Besides, if $c = 1$, formula (5.4) reduces to (5.3). So in the sequel only the proof of (5.5) will be given. First we note that for each $\pi \in \mathscr{D}_{\mathbf{n}}$, we have

$$\operatorname{exc}\pi + \operatorname{dec}\pi = n_1 + \ldots + n_m.$$

Hence, in (5.4) if we substitute c by $(\delta+i)/(\delta-i)$, β by η, and x_k by $(\delta-i)^k$, we get immediately

$$\sum_{\mathbf{n}\geq 0} \Big(\sum_{\pi\in\mathscr{D}_{\mathbf{n}}} (\delta-i)^{\operatorname{exc}\pi}(\delta+i)^{\operatorname{dec}\pi}\eta^{\operatorname{cyc}\pi}\Big) \frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!}$$
$$= \Big\{1 - \frac{\delta^2+1}{2i} \sum_{k=2}^{m} [(\delta+i)^{k-1} - (\delta-i)^{k-1}]e_k\Big\}^{-\eta},$$

which yields (5.5) after simplification.

In view of the importance of (5.4), we give here two more proofs of this formula.

LEMMA 7. *Let $\Phi(x_1, x_2, \ldots)$ be a power series in $x_1, x_2, \ldots$. Then*

$$[\mathbf{x}]\Phi(e_1, e_2, \ldots) = \Big[\frac{x^m}{m!}\Big]\Phi\Big(\frac{x^1}{1!}, \frac{x^2}{2!}, \ldots\Big).$$

For a proof of this result, we refer the reader to [**27**, p. 233].

First new proof of (5.4). We will count the c-derangements through the wave segment decomposition given in Proposition 4. For $1 \leq i \leq m-1$, let $\mathbf{n}_i = (n_1, \ldots, n_i)$ and W_i be the set of wave arrangements of $[n_1] \cup \ldots \cup [n_i]$. Let

$\pi = a_1 \ldots a_n$ be a wave arrangement of W_i and $\alpha \in \mathbb{N}_*$ be an element with colour $c(\alpha) = i+1$. It is then clear that $\alpha\pi = \alpha a_1 \ldots a_n$ is a wave segment beginning with colour $i+1$. Define the generating function for W_i by

$$f_i(x_1, \ldots, x_i ; \gamma) = \sum_{\mathbf{n}_i} \Big(\sum_{\pi \in W_i} \gamma^{\operatorname{dec} \pi} \Big) \frac{x_1^{n_1}}{n_i!} \cdots \frac{x_m^{n_m}}{n_m!}. \tag{5.6}$$

It follows that the generating function of the wave segments beginning with colour $i+1$ is $\gamma x_{i+1} f_i(x_1, \ldots, x_i ; \gamma)$. Recall that each c-derangement π corresponds bijectively to a $(m-1)$-tuple $((\Omega_2, \sigma_2), \ldots, (\Omega_m, \sigma_m))$. Since the generating function of permutations with each cycle being weighted by η is $(1-x)^{-\eta}$, the generating function of $(\Omega_{i+1}, \sigma_{i+1})$ should be $(1 - \gamma x_{i+1} f_i(x_1, \ldots, x_i ; \gamma))^{-\eta}$ where $1 \leqslant i \leqslant m-1$. Thanks to Proposition 4, we obtain

$$\sum_{\mathbf{n} \geqslant 0} D(\mathbf{n} ; \gamma, \eta) \frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!} = \prod_{i=1}^{m-1} (1 - \gamma x_{i+1} f_i(x_1, \ldots, x_i ; \gamma))^{-\eta}. \tag{5.7}$$

On the other hand, in view of Corollary 3, both sides of (5.7) are symmetric functions of $x_1, \ldots, x_m$, which implies that the function

$$\prod_{i=1}^{m-1} (1 - \gamma x_{i+1} f_i(x_1, \ldots, x_i ; \gamma)) \tag{5.8}$$

is also symmetric. Furthermore, we note that (5.8) is actually *linear* in x_m. By symmetry, it is a *linear symmetric function* of $x_1, \ldots, x_m$. So we can expand (5.8) as a linear combination of the elementary symmetric functions e_i $(0 \leqslant i \leqslant m)$ (cf. [**33**]):

$$\prod_{i=1}^{m-1} (1 - \gamma x_{i+1} f_i(x_1, \ldots, x_i ; \gamma)) = a_0 + a_1 e_1 + \ldots + a_m e_m, \tag{5.9}$$

where the coefficients a_i $(0 \leqslant i \leqslant m)$ are to be determined. Using (5.9) we can rewrite (5.7) as

$$\sum_{\mathbf{n} \geqslant 0} D(\mathbf{n} ; \gamma, \eta) \frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!} = (a_0 + a_1 e_1 + \ldots + a_m e_m)^{-\eta}. \tag{5.10}$$

Applying Lemma 7 to (5.10) and noticing that the coefficient of $x_1 \ldots x_m$ in the left-hand side of (5.10) is the generating polynomial $d_m(\gamma, \eta)$ of the ordinary derangements of $\mathscr{D}_m$ (cf. Proposition 2), we obtain the following equation:

$$\Big(\sum_{k \geqslant 0} a_k \frac{x^k}{k!} \Big)^{-\eta} = \Big\{ 1 - \sum_{k \geqslant 2} (\gamma + \gamma^2 + \ldots + \gamma^{k-1}) \frac{x^k}{k!} \Big\}^{-\eta}.$$

Equating the coefficients of x^n gives $a_0 = 1$, $a_1 = 0$ and, for $k \geqslant 2$,

$$a_k = -(\gamma + \gamma^2 + \ldots + \gamma^{k-1}).$$

Finally, substituting the above values into (5.10) yields (5.4).

Second new proof of (5.4). Let $f_i(\gamma) = f_i(x_1, \ldots, x_i ; \gamma)$; it is then well known [**27**, p. 78] that

$$f_i(\gamma) = \frac{\prod_{j=1}^i (1 + \gamma x_j) - \prod_{j=1}^i (1 + x_j)}{\gamma \prod_{j=1}^i (1 + x_j) - \prod_{j=1}^i (1 + \gamma x_j)}$$

where $1 \leqslant i \leqslant m-1$. It follows that

$$(5.11) \qquad (1-\gamma x_{i+1} f_i(\gamma))^{-1} = \frac{\gamma \prod_{j=1}^{i} (1+x_j) - \prod_{j=1}^{i} (1+\gamma x_j)}{\gamma \prod_{j=1}^{i+1} (1+x_j) - \prod_{j=1}^{i+1} (1+\gamma x_j)}.$$

Substituting this in (5.7) we get

$$\begin{aligned} \sum_{\mathbf{n} \geqslant 0} D(\mathbf{n}; \gamma, \eta) \frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!} &= \prod_{i=1}^{m-1} (1-\gamma x_{i+1} f_i(\gamma))^{-\beta} \\ &= \prod_{i=1}^{m-1} \left(\frac{\gamma \prod_{j=1}^{i} (1+x_j) - \prod_{j=1}^{i} (1+\gamma x_j)}{\gamma \prod_{j=1}^{i+1} (1+x_j) - \prod_{j=1}^{i+1} (1+\gamma x_j)} \right)^{-\beta} \\ &= \left\{ \frac{\gamma(1+x_1) - (1+\gamma x_1)}{\gamma \prod_{j=1}^{m} (1+x_j) - \prod_{j=1}^{m} (1+\gamma x_j)} \right\}^{-\beta} \\ &= (1-\gamma e_2 - (\gamma+\gamma^2) e_3 - \ldots - (\gamma + \ldots + \gamma^{m-1}) e_m)^{-\beta}, \end{aligned}$$

as desired.

6. *Combinatorial interpretations of the linearization coefficients*

Combining Theorems 2 and 3, we immediately derive the following combinatorial interpretations of the linearization coefficients of the orthogonal Sheffer polynomials.

THEOREM 4. *The following identities hold:*

$$(6.1) \qquad \mathcal{H}(\mathbf{n}) = |\mathcal{I}_{\mathbf{n}}|;$$

$$(6.2) \qquad \mathcal{C}(\mathbf{n}; a) = \sum_{\pi} a^{\operatorname{bloc} \pi} \quad (\pi \in \mathcal{P}_{\mathbf{n}});$$

$$(6.3) \qquad \mathcal{L}(\mathbf{n}; \alpha) = \sum_{\pi} (1+\alpha)^{\operatorname{cyc} \pi} \quad (\pi \in \mathcal{D}_{\mathbf{n}});$$

$$(6.4) \qquad \mathcal{M}(\mathbf{n}; \beta, c) = \sum_{\pi} c^{\operatorname{exc} \pi} \beta^{\operatorname{cyc} \pi} \quad (\pi \in \mathcal{D}_{\mathbf{n}});$$

$$(6.5) \qquad \mathcal{P}(\mathbf{n}; \delta, \eta) = \sum_{\pi} (\delta + i)^{\operatorname{dec} \pi} (\delta - i)^{\operatorname{exc} \pi} \eta^{\operatorname{cyc} \pi} \quad (\pi \in \mathcal{D}_{\mathbf{n}}).$$

As mentioned in § 1, formulae (6.1)–(6.4) have been established by other methods in [**8**, **14**, **20**, **41**]. One notices that Theorem 4 shows that the linearization coefficients of Hermite, Charlier, Laguerre and Meixner polynomials are actually the polynomials of some appropriate variables with non-negative integral coefficients. Therefore, the *classical positivities* (cf. [**4**]) of these coefficients become *obvious*. However, although it is evident from (6.5) that $\mathcal{P}(\mathbf{n}; \delta, \eta)$ is a polynomial of δ and η, the fact that the coefficients are *non-negative integers* is not obvious.

PROPOSITION 5. *The coefficients of the polynomial $\mathcal{P}(\mathbf{n}; \delta, \eta)$ of δ and η are non-negative integers.*

Proof. We first prove that it is a polynomial with integral coefficients. Clearly, it is sufficient to prove that in the summation of (6.5) each term appears with its conjugate. For this end, we construct an involution T on $\mathcal{D}_{\mathbf{n}}$ as follows. For each $\pi \in \mathcal{D}_{\mathbf{n}}$, we define $T(\pi) = \pi'$ by $\pi'(a) = b$ if and only if $\pi(b) = a$, where a, b are elements of $[\mathbf{n}]$. It is easy to see that T is an involution without fixed point on $\mathcal{D}_{\mathbf{n}}$ such that

$$\text{cyc}\,\pi = \text{cyc}\,\pi', \quad \text{dec}\,\pi = \text{exc}\,\pi', \quad \text{exc}\,\pi = \text{dec}\,\pi'. \tag{6.6}$$

Therefore, in the summation of (6.5), the summand corresponding to π' is the *conjugate* of that corresponding to π'. It follows that $\mathcal{P}(\mathbf{n}\,;\delta,\eta)$ is a polynomial in δ and η with integral coefficients. It remains to prove that the coefficients are positive. We shall prove this for $m \leqslant 4$, for the general cases are similar to the case where $m = 4$. Set $P_n := P_n(x\,;\delta,\eta)$. By (3.22) we have

$$\varphi(P_{n_1}P_{n_2}P_{n_3}) = \sum_{s\geqslant 0} \frac{n_1!n_2!n_3!(\eta)_s(\delta^2+1)^s(2\delta)^{n_1+n_2+n_3-2s}}{(s-n_1)!(s-n_2)!(s-n_3)!(n_1+n_2+n_3-2s)!}. \tag{6.7}$$

Therefore, the proposition is true for $m \leqslant 3$. For $m = 4$ we derive from the linearization formula (1.0) that

$$\varphi(P_{n_1}P_{n_2}P_{n_3}P_{n_4}) = \sum_{k\geqslant 0} \frac{\varphi(P_kP_{n_1}P_{n_2})\varphi(P_kP_{n_3}P_{n_4})}{\varphi(P_kP_k)}. \tag{6.8}$$

By (6.7) and noting that $\varphi(P_kP_k) = (k!)^2(\eta)_k(\delta^2+1)^k$, we may conclude that (6.8) is a polynomial with positive coefficients.

When $\delta = 0$, the generating function (1.7) reduces to

$$\sum_{n\geqslant 0} M_n(x\,;0,\eta)\frac{t^n}{n!} = (1+t^2)^{-\eta/2}\exp(x \arctan t). \tag{6.9}$$

Note that the polynomials $i^nM_n(-ix\,;0,\eta)$ were also introduced by Carlitz [**11**]. We first derive some consequences from Theorem 4 for $\mathcal{P}(\mathbf{n}\,;0,\eta)$ before we proceed to give another interesting interpretation in a different context.

PROPOSITION 6. *If $n = n_1 + \ldots + n_m$ is odd, then $\mathcal{P}(\mathbf{n}\,;0,\eta) = 0$.*

Proof. Recall that the involution T defined in the proof of Proposition 5 has the following properties: if $T(\pi) = \pi'$, then

$$\text{cyc}\,\pi = \text{cyc}\,\pi', \quad \text{exc}\,\pi = n - \text{exc}\,\pi'. \tag{6.10}$$

Whence, if $n = n_1 + \ldots + n_m$ is odd, in the sum of (6.5) the weight of π is killed by that of π', that is,

$$(-1)^{\text{exc}\,\pi}i^n\eta^{\text{cyc}\,\pi} + (-1)^{\text{exc}\,\pi'}i^n\eta^{\text{cyc}\,\pi'} = 0.$$

Thus the total sum (6.5) is clearly zero.

Let $\mathcal{E}_{\mathbf{n}}$ be the set of permutations of $\mathcal{D}_{\mathbf{n}}$ of which all the cycles are of even length. It follows that $\mathcal{E}_{\mathbf{n}} = \varnothing$ if $n = n_1 + \ldots + n_m$ is odd.

PROPOSITION 7. *We have*

$$\mathcal{P}(\mathbf{n}\,;0,\eta) = \sum_{\pi\in\mathcal{E}_n} (-1)^{\text{exc}\,\pi}i^n\eta^{\text{cyc}\,\pi}. \tag{6.11}$$

Proof. Clearly, if we could define an involution θ: $\mathcal{D}_{\mathbf{n}}\setminus\mathcal{E}_{\mathbf{n}} \to \mathcal{D}_{\mathbf{n}}\setminus\mathcal{E}_{\mathbf{n}}$ such that

$\theta(\pi) = \pi'$ with

(6.12) $\qquad \operatorname{cyc} \pi = \operatorname{cyc} \pi'$ and $\operatorname{exc} \pi + \operatorname{exc} \pi'$ odd,

the identity (6.10) would follow immediately. For each $\pi \in \mathscr{D}_{\mathbf{n}} \setminus \mathscr{E}_{\mathbf{n}}$, we may assume that the set of cycles of π is totaly ordered (for example, according to the order of the highest ranking element of each cycle). Among the cycles of π, let $a_1 \to a_2 \to \ldots \to a_r$ be the least cycle with odd length r. Then $\pi' = \theta(\pi)$ is defined by the same cycles as π, except that the last cycle of odd length is $a_r \to a_{r-1} \to \ldots \to a_1$. Clearly θ is an involution on $\mathscr{D}_{\mathbf{n}} \setminus \mathscr{E}_{\mathbf{n}}$ and satisfies (6.12), for $\operatorname{exc} \pi + \operatorname{exc} \pi' \equiv r \equiv 1 \pmod 2$.

The new interpretation of $\mathscr{P}(\mathbf{n}; 0, \eta)$ relies on the model of *generalized up-down permutations*. We consider a *permutation* of a subset A of $\mathbb{N}_*$ as a linear arrangement $a_1 a_2 \ldots a_n$ of the elements of A. A permutation $\pi = a_1 a_2 \ldots a_n$ is said to be *up-down* if $c(a_1) < c(a_2) \geqslant c(a_3) < \ldots \lesseqqgtr c(a_n)$, and *basic* if it begins with its smallest element. Also, we say that the *length* of the permutation $\pi = a_1 a_2 \ldots a_n$ is n. Let $\mathscr{S}_{\mathbf{n}}$ be the set of all permutations on $[\mathbf{n}]$ and $\mathscr{U}_{\mathbf{n}}$ the set of up-down permutations on $[\mathbf{n}]$. A *record* of a permutation $\pi = a_1 a_2 \ldots a_n$ is an a_j such that $i < j$ implies $a_i > a_j$. Denote by $\operatorname{rec} \pi$ the number of records of π and by $F(\pi)$ the first element of π. A permutation $\pi = a_1 a_2 \ldots a_n$ is *basic* if it begins with its smallest element. We let $\mathscr{B}_{\mathbf{n}}$ be the set of basic permutations of $\mathscr{S}_{\mathbf{n}}$.

As noted by Gessel [24], every permutation $\pi = a_1 a_2 \ldots a_n$ has a unique factorization $\pi = \beta_1 \beta_2 \ldots \beta_k$ such that each β_i is basic and $F(\beta_1) > F(\beta_2) > \ldots > F(\beta_k)$. Such a factorization is called a *basic decomposition* of π and the β_i the *basic components* of π. We call a permutation *reduced* if it is a permutation of $[\mathbf{n}]$. To any permutation $\pi = a_1 a_2 \ldots a_n$ we may associate a reduced permutation, $\operatorname{red}(\pi)$, by replacing in π the jth smallest element of colour i by (i, j). Thus

$$\operatorname{red}((1,7)(2,9)(1,2)(2,6)) = (1,2)(2,2)(1,1)(2,1).$$

The *content* of the permutation $\pi = a_1 a_2 \ldots a_n$ is the set $\operatorname{con}(\pi) = \{a_1, a_2, \ldots, a_n\}$. We note that a permutation is determined by its reduction and its content.

A function ω defined on permutations (with values in some commutative algebra over the rationals) is *multiplicative* if for all permutations π:

(1) $\omega(\pi) = \omega(\operatorname{red}(\pi))$;

(2) if $\beta_1 \beta_2 \ldots \beta_k$ is the basic decomposition of π, then

$$\omega(\pi) = \omega(\beta_1)\omega(\beta_2) \ldots \omega(\beta_k).$$

Thus a multiplicative function is determined by its values on reduced basic permutations, and these may be chosen arbitrarily. One of the fundamental properties of a multiplicative function is the so-called exponential formula.

LEMMA 8. *Let ω be a multiplicative function on permutations. Set $g_{\mathbf{n}} = \sum_{\pi \in \mathscr{S}_{\mathbf{n}}} \omega(\pi)$ and $f_{\mathbf{n}} = \sum_{\pi \in \mathscr{B}_{\mathbf{n}}} \omega(\pi)$. Then we have*

$$\sum_{\mathbf{n} \geqslant 0} g_{\mathbf{n}} \frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!} = \exp \sum_{\mathbf{n} > 0} f_{\mathbf{n}} \frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!}.$$

We define the weight of a permutation π of $[\mathbf{n}]$ by $w(\pi) = \eta^{\operatorname{rec}(\pi)}$ if π is up-down and of even length and $w(\pi) = 0$ otherwise. Obviously $S_{\mathbf{n}}(\eta) = \sum_{\pi \in \mathscr{S}_{\mathbf{n}}} w(\pi)$

and $B_{\mathbf{n}}(\eta)=\sum_{\pi\in\mathscr{B}_{\mathbf{n}}} w(\pi)$ are, respectively, the generating functions of up-down and basic up-down permutations of even length.

LEMMA 9. *We have*

$$\sum_{\mathbf{n}\geq 0} S_{\mathbf{n}}(\eta)\frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!}=\Big(\sum_{n\geq 0}(-1)^n e_{2n}\Big)^{-\eta}. \tag{6.13}$$

Proof. As the weight $w(\pi)$ is multiplicative, by Lemma 8, we have

$$\sum_{\mathbf{n}\geq 0} S_{\mathbf{n}}(\eta)\frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!}=\exp\sum_{\mathbf{n}>0} B_{\mathbf{n}}(\eta)\frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!}=\exp\eta\sum_{\mathbf{n}>0} B_{\mathbf{n}}(1)\frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!}. \tag{6.14}$$

But it is well known [27, p. 234] that

$$\exp\sum_{\mathbf{n}>0} B_{\mathbf{n}}(1)\frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}}=\sum_{\mathbf{n}\geq 0} S_{\mathbf{n}}(1)\frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!}=\Big(\sum_{n\geq 0}(-1)^n e_{2n}\Big)^{-1}.$$

Then, the proof is completed by putting the last identity in (6.14).

THEOREM 5. *For any* $\mathbf{n}=(n_1,\dots,n_m)$, *the polynomials* $\mathscr{P}(\mathbf{n};0,\eta)$ *are the generating polynomials of up-down permutations of even length on* $[\mathbf{n}]$ *weighted by records, viz.,*

$$\mathscr{P}(\mathbf{n};0,\eta)=\begin{cases}\sum_{\pi\in\mathscr{U}_{\mathbf{n}}}\eta^{\operatorname{rec}\pi} & \text{if } n_1+\dots+n_m \text{ is even,}\\ 0 & \text{if } n_1+\dots+n_m \text{ is odd.}\end{cases}$$

Proof. If $\delta=0$, the right-hand side of (3.13) reduces to that of (6.13). The theorem follows then from Lemma 9 and Theorem 2.

By comparison of Proposition 6 with Theorem 5, we derive the following

COROLLARY 4. *We have*

$$\sum_{\pi\in\mathscr{C}_{\mathbf{n}}}(-1)^{\operatorname{exc}\pi}i^n\eta^{\operatorname{cyc}\pi}=\sum_{\pi\in\mathscr{U}_{\mathbf{n}}}\eta^{\operatorname{rec}\pi}. \tag{6.15}$$

REMARK. When $\mathbf{n}=(1,1,\dots,1)$ and $\eta=1$, identity (6.15) reduces to a well-known result [43, Theorem 4.7].

In the case where $\eta=0$, the polynomials $p_n(x):=p_n(x;0,0)$ are no longer orthogonal with respect to φ. However, there is another linear functional ζ of some interest, which may be defined by the generating function of the sequence $\zeta(x^n)$ $(n\geq 0)$:

$$\sum_{n\geq 0}\zeta(x^n)\frac{x^n}{n!}=\tan x. \tag{6.16}$$

It then follows from Lemma 5 that

$$\zeta\Big(\exp x\sum_{k=1}^{m}\arctan x_k\Big)=\frac{\sum_{n\geq 0}(-1)^n e_{2n+1}}{\sum_{n\geq 0}(-1)^n e_{2n}}.$$

In other words, the following identity holds:

$$\sum_{\mathbf{n}}\zeta\Big(\prod_{k=1}^{m}p_{n_k}(x)\Big)\frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!}=\frac{\sum_{n\geq 0}(-1)^n e_{2n+1}}{\sum_{n\geq 0}(-1)^n e_{2n}}. \tag{6.17}$$

However, it is well known that the right-hand side of (6.17) is the generating function of the number of up-down permutations of *odd* length on [**n**] (cf. [**27**, p. 196]). Consequently, $\zeta(\prod_{k=1}^{m} p_n(x))$ is the number of up-down permutations of *odd* length on [**n**]. Hence, in particular, we have the quasi-orthogonality

$$\zeta(p_n(x)p_m(x)) = \begin{cases} n!\,m! & \text{if } |n-m| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

These results may be of some interest in the theory of *coefficient extraction* of symmetric functions (cf. [**23**]).

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Département de Mathématique
Université Louis-Pasteur
7 rue René-Descartes
F-67084 Strasbourg
France